Three Loop Quark Number Susceptibilities in Hard Thermal Loop approximation

Aritra Bandyopadhyay

Theory Divison
Saha Institute of Nuclear Physics
Kolkata, India

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Outline

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Motivations

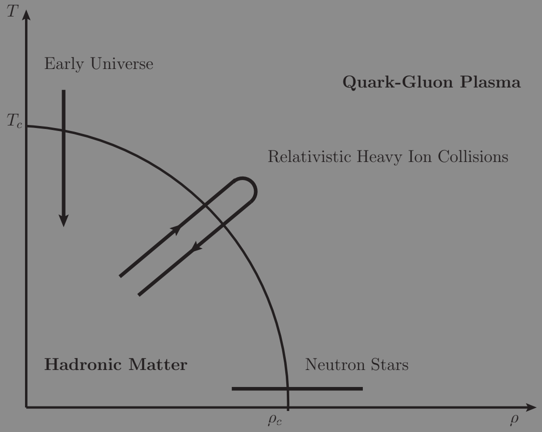

Figure: Phase diagram of Nuclear Matter

Goal

Equation of State of QCD matter
at finite temperature and finite quark chemical potential
$\Downarrow$
Generic quantity in studying various thermodynamic quantities
and required for hydrodynamic evolution of QGP
$\Downarrow$
Quark number susceptibilities
$\Downarrow$
Number fluctuations
$\Downarrow$
Charge fluctuations

Outline

Methods of study

- Lattice gauge theory $\Longrightarrow$ non-perturbative, reliable method.
- But it has some insufficiencies, mostly noticed at nonzero chemical potential (infamous sign problem).
- So, some alternative method is needed to study QCD thermodynamics at finite temperature and finite quark chemical potential.

Drawbacks of Bare perturbation theory

- Unfortunately, unless the coupling constant is extremely small, we cannot use bare perturbation theory.
- But at phenomenologically accessible temperatures near T_c for the QCD deconfinement phase transition, $g \approx 2$.
- Resulting weak coupling expansion, truncated order by order in g is poorly convergent.
- Even at temperature many order of magnitude higher than those achievable in heavy ion collisions $\Longrightarrow$ Not of any quantitative use.

Drawbacks of Bare perturbation theory contd..

- Evaluating two-loop scalar mass correction infers that Bare Perturbative QCD is incomplete and gradually more and more infrared divergent.

Drawbacks of Bare perturbation theory contd..

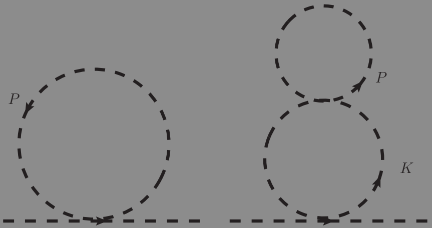

Figure: Scalar one and two-loop self energy diagrams

- Considering self interacting ϕ^4 theory, one loop self energy is finite ($\Pi_1 = \frac{g^2T^2}{24}$) but the two loop self energy becomes infrared divergent.

$$\Pi_2 = \frac{1}{2} g^2 \left(\frac{g^2T^2}{24} \right) \sum\!\!\!\!\!\!\!\!\int_K \frac{1}{K^4}$$

Drawbacks of Bare perturbation theory contd..

- Evaluating two-loop scalar mass correction infers that Bare Perturbative QCD is incomplete and gradually more and more infrared divergent.

- It also has some serious problems like the apparent gauge dependence of the gluon damping rate.

  Not only in magnitude, but also in sign.

$$\gamma_g = a\frac{g^2T}{8\pi}$$

$a = 1$ for Coulomb Gauge
$a = -5$ for Feynmann Gauge.

Hard Thermal Loop perturbation theory

- We therefore need methods which can provide reliable approximations to QCD thermodynamics at moderate coupling.
- In hot gauge theories, the usual connection between the order of loop expansion and powers of g is lost. Effects of leading order in g arise from every order in the loop expansion.
- In 1989 Pisarski and Braaten argued that, all we need is to resum this infinite subset of feynman graphs in order to get a consistent perturbative expansion in g, which is the basic of Hard Thermal Loop (HTL) perturbation theory.
  [Phys. Rev. Lett. 63 (1989) 1129 ; Nucl. Phys. B 337(1990) 569-634]

Hard Thermal Loops

- Two natural momentum($P = (\omega, \vec{p})$) scales in hot field theories are termed as soft if ω and p are of order gT, and hard if either is of order T.

- Hard Thermal Loops arise from integration regions, in which the loop momentum is hard, while all the external momenta are soft.

$$\text{Loop corrections } = \frac{g^2T^2}{P^2} \times \text{ corresponding Tree level amplitude}$$

- So, when the external momentum P is soft ($\approx gT$), diagrams who are higher order in loop expansion, are as important as the tree level diagrams.

Effective Propagator

- Now for soft legs, it's necessary to use an effective propagator, in which all the hard thermal loops are resummed into.

- Now, if we resum the whole series, then the effective propagator will be,

$$\Delta^*(P) = \frac{i}{(P^2 - \Pi(P))} = \frac{i}{(P^2 - g^2T^2)}$$

- Similarly, when every line joining a vertex is soft, an effective vertex is used, through Ward Identity.

Outline

Thermodynamic pressure of massless QCD at $T, \mu \neq 0$

- Thermodynamic observables via partition function in path integral representation is given by,

$$\mathcal{Z}(T,\mu) \equiv \mathrm{Tr}(e^{-\beta H}) \to \int D\mathcal{A}\; D\psi D\bar{\psi}\, \exp[-S_E]$$

$$S_E = \int_0^\beta d\tau \int d^{d-1}x(\mathcal{L}_E - \mu\mathcal{N})$$

- So, thermodynamic pressure can be determined as,

$$P = \mathrm{lim}_{V\to\infty} \frac{T}{V} \ln \mathcal{Z}$$

- Other quantities can also be evaluated using thermodynamic relations.

Free case

- Using the idea of functional determinant,

$$\int D\psi D\bar{\psi} \exp[i \int d^4x \bar{\psi}(i\partial\!\!\!/ - m)\psi] \;=\; \det(i\partial\!\!\!/ - m)$$

- So for the free case, the one loop quark free energy of a hot quark-gluon plasma with finite quark chemical potential comes out to be,

$$\begin{aligned} F_q &= -\int \frac{d^4K}{(2\pi)^4} \ln[\det(K\!\!\!/)] \\ &= -\left[\frac{7\pi^2}{180}T^4 + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2}\right] \end{aligned}$$

Leading order HTLpt case

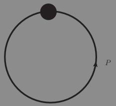

Figure: Leading order HTL quark free energy

$$F_q^{HTL} = -\int \frac{d^4K}{(2\pi)^4} \ln[\det(\not{K} - \Sigma(K))$$

- Assuming the temperature is much greater than the mass parameters m_q, one-loop free energies can be expanded in powers of m_q.

$$F_q = \underbrace{-\left[\frac{7\pi^2}{180}T^4 + \frac{\mu^2T^2}{6} + \frac{\mu^4}{12\pi^2}\right]}_{\text{Free part}} + \underbrace{\frac{1}{6}\left[T^2 + \frac{3\mu^2}{\pi^2}\right]m_q^2 + \frac{(\pi^2-6)}{12\pi^2}m_q^4}_{\text{HTL Correction}}$$

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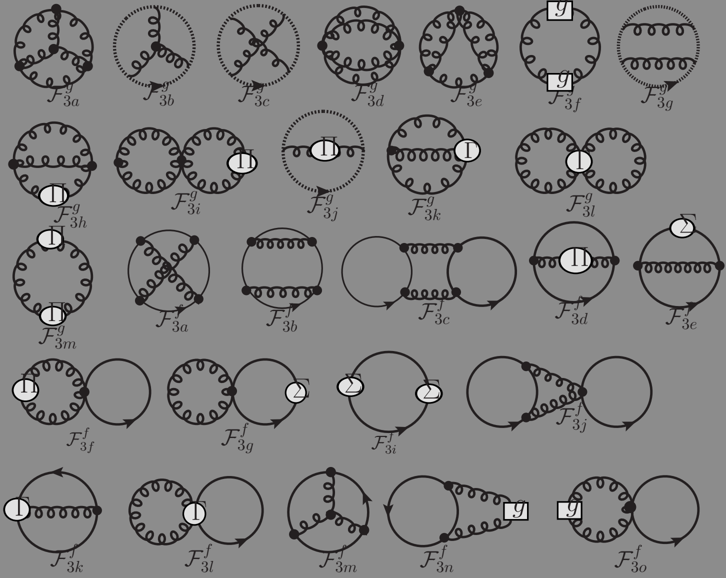
Messages from Two-loop HTLpt

- Resummation causes overcounting. Unlike pQCD loop and coupling expansion are not symmetrical in HTLpt. Higher loops contribute to lower loop order.
- NLO calculation corrects the overcounting in LO, but the overall problem of overcounting still remains.
- Two loop HTLpt is a modest improvement over pQCD in terms of convergence and sensitivity of the renormalization scale.
- A Three loop calculation is essential to cure the overcounting/ convergence problem in Two loop order.

Why Three loop?

- As in HTLpt we use moderate value of the coupling constant g, so truncating higher order terms is not very much useful, as it also has significant importance.
- Now, there has to be a definite limit to how far perturbation theory needs to be pushed before the infrared divergence deny further analytic progress.
- In our case, this infrared problem(Logarithmic) is met at the 4-loop order.
- So, a complete Three loop calculation of the pressure is the best one can do with recent machinary.

Three loop diagrams



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Quark Number Susceptibilities

- Quark Number Susceptibilities can be defined by Taylor expanding the pressure as a function of quark chemical potentials and temperature,

$$\frac{\mathcal{P}}{T^4} = \sum_{ijk\cdots} \frac{1}{i!j!k!\cdots} \chi^{uds\cdots}_{ijk\cdots} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \cdots$$

$$\chi^{uds\cdots}_{ijk\cdots}(T) \equiv \left.\frac{\partial^{i+j+k+\cdots}\mathcal{P}(T,\mu)}{\partial\mu_u^i\partial\mu_d^j\partial\mu_s^k\cdots}\right|_{\mu=0}$$

- Similarly it can also be represented in terms of chemical potential corresponding to Baryon number B, electric charge Q and strangeness S of hadrons.

$$\chi^{BQS\cdots}_{ijk\cdots}(T) \equiv \left.\frac{\partial^{i+j+k+\cdots}\mathcal{P}(T,\mu)}{\partial\mu_B^i\partial\mu_Q^j\partial\mu_S^k\cdots}\right|_{\mu=0}$$

Baryon Number Susceptibility

- The n^{th} order Baryon number susceptibility is defined as,

$$\chi_B^n(T) \equiv \left.\frac{\partial^n \mathcal{P}}{\partial \mu_B^n}\right|_{\mu_B=0}$$

- For a three flavor system (u, d, s), quark chemical potentials are related to the chemical potential of baryon number, strangeness and electric charge as,

$$\begin{aligned}
\mu_u &= \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \\
\mu_d &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \\
\mu_s &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S
\end{aligned}$$

BNS to QNS

- So for a three flavor system (u, d, s), the baryon number susceptibilities can be related to the quark number susceptibilities,

$$
\begin{aligned}
\chi_2^B &= \frac{1}{9}\left[\chi_2^{uu} + \chi_2^{dd} + \chi_2^{ss} + 2\chi_2^{ud} + 2\chi_2^{ds} + 2\chi_2^{us}\right] \\
\chi_4^B &= \frac{1}{81}\left[\chi_4^{uuuu} + \chi_4^{dddd} + \chi_4^{ssss} + 4\chi_4^{uuud} + 4\chi_4^{uuus} + 4\chi_4^{dddu}\right. \\
&\quad +4\chi_4^{ddds} + 4\chi_4^{sssu} + 4\chi_4^{sssd} + 6\chi_4^{uudd} + 6\chi_4^{uuss} + 6\chi_4^{ddss} \\
&\quad \left. +12\chi_4^{uuds} + 12\chi_4^{ddus} + 12\chi_4^{ssud}\right]
\end{aligned}
$$

- If we treat all the quarks as having the same chemical potential $\mu_u = \mu_d = \mu_s = \frac{1}{3}\mu_B = \mu$, then the above relations reduce to,

$$
\begin{aligned}
\chi_2^B &= \chi_2^{uu} \\
\chi_4^B &= \chi_4^{uuuu}
\end{aligned}
$$

Diagonal and Off-diagonal QNS

- To evaluate QNS from the general expression of the NNLO thermodynamic potential, different quark chemical potentials are used.
- So, the resulting susceptibilities can be either
  diagonal $\Longrightarrow$ same flavor on all derivatives, or
  off-diagonal $\Longrightarrow$ different flavor on all/some derivatives.
- But for the off-diagonal case most of the diagrams do not contribute because they do not have two separate quark loops.

Contributing diagrams

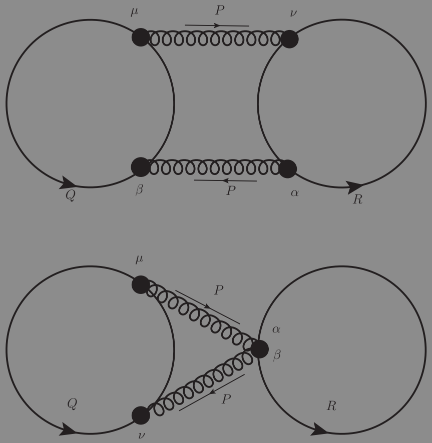

Figure: NNLO Feynman diagrams that mainly contribute to off-diagonal susceptibility

Outline

NNLO HTLpt Pressure

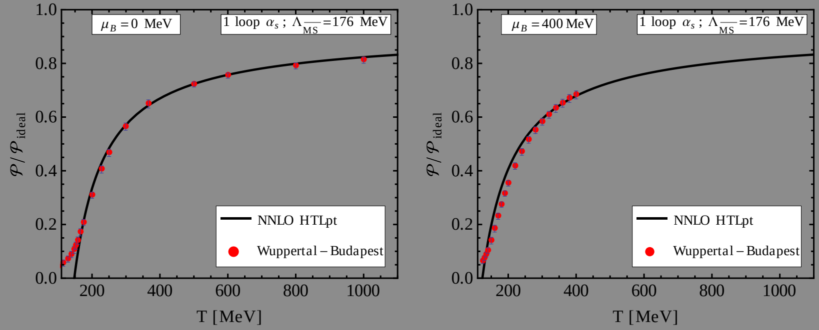

Figure: Comparison of NNLO HTLpt pressure with lattice data $(N_f = 3)$ for $\mu_B = 0$ (left) and $\mu_B = 400 MeV$ (right)

Second order QNS

- At second order all off-diagonal susceptibilities vanishes in HTLpt.

$$\chi_2^{ud} = \chi_2^{ds} = \chi_2^{us} = 0$$

- So, the diagonal susceptibilities are directly proportional to the Baryon number susceptibility.

$$\chi_2^{uu} = \chi_2^{dd} = \chi_2^{ss} = \frac{1}{3}\chi_2^{B}$$

Second order QNS plot

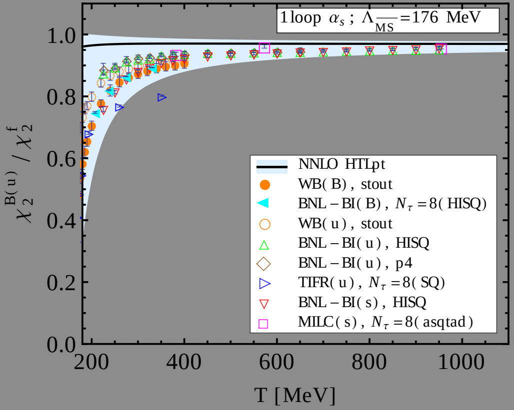

Figure: Comparison of NNLO HTLpt second order Quark Number Susceptibility with lattice data. $(N_f = 3)$

Why Band?

QCD running coupling:

$$\alpha_s(\Lambda) = \frac{g(\Lambda)^2}{4\pi} = \frac{2\pi}{\left(11 - \frac{2}{3}N_f\right)\log\left(\Lambda/\Lambda_{\rm QCD}\right)},$$

- $\Lambda_{\rm QCD} \rightarrow$ QCD scale.
- $\Lambda \rightarrow$ Energy scale.
- The middle line corresponds to $\Lambda = 2\pi T =$ Lowest order Matsubara at finite temperature.
- Band$\rightarrow$ $\Lambda \rightarrow (\pi T - 4\pi T)$

Fourth order QNS

- At fourth order only one type of off-diagonal susceptibilities survive, namely,

$$\chi_4^{uudd} = \chi_4^{ddss} = \chi_4^{uuss}$$

which is related to the diagonal susceptibility as,

$$\chi_4^{uuuu} = 27\chi_4^{B} - 6\chi_4^{uudd}$$

Fourth order Diagonal QNS plot

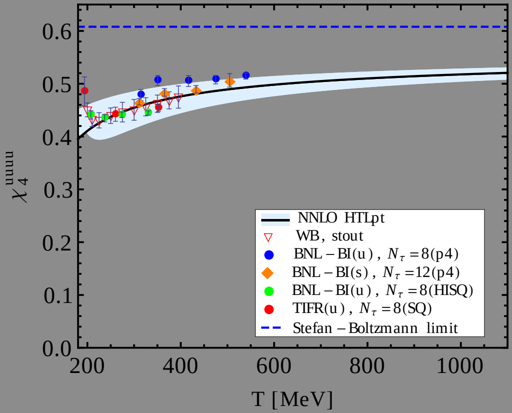

Figure: Comparison of NNLO HTLpt fourth order diagonal Quark Number Susceptibility with lattice data. $(N_f = 3)$ The dashed line indicates the Stefan-Boltzmann limit for this quantity.

Fourth order Off-diagonal QNS plot

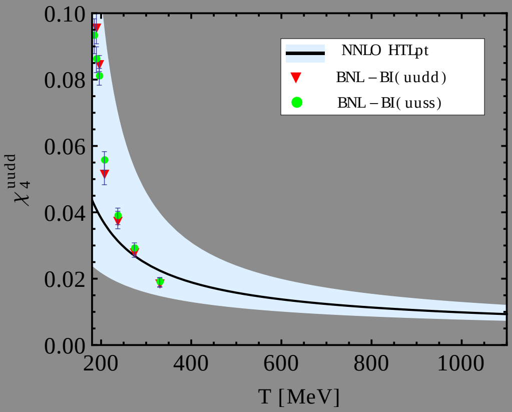

Figure: Comparison of NNLO HTLpt fourth order off-diagonal Quark Number Susceptibility with lattice data. ($N_f = 3$)

Ratio of the fourth to second order QNS

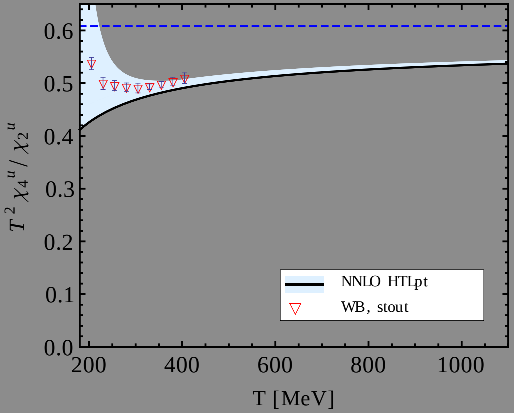

Figure: Comparison of NNLO HTLpt ratio of the fourth to second order diagonal Quark Number Susceptibility with lattice data. $(N_f = 3)$

Sixth order diagonal and off-diagonal QNS

- In sixth order, alongside diagonal QNS χ_{600} there are two types of non-vanishing off-diagonal QNS, namely χ_{420} and χ_{222}.

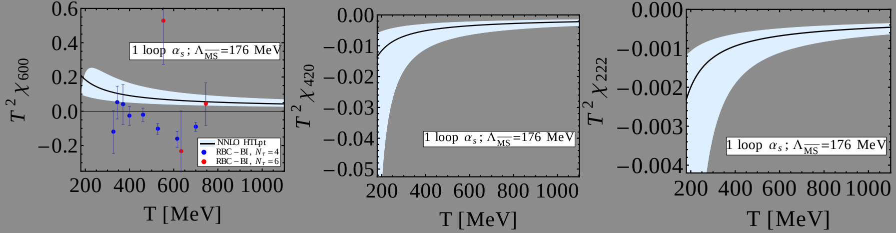

Figure: NNLO HTLpt scaled sixth-order diagonal and off-diagonal QNS, χ_{600}(left), χ_{420}(middle) and χ_{222}(right) as a function of temperature.($N_f = 3$)

Outline

Conclusions and Outlook

- NNLO Quark Number Susceptibilities can be used as an input in further calculations of charge fluctuations, which can be used as a signal for Quark Gluon Plasma.
- As in all the cases HTLpt results are in good agreement with the available lattice data, it would therefore be challenging to apply Three Loop HTLpt to the calculation of dynamic quantities.

Collaborators

Najmul Haque
SINP

J.O. Andersen
NTNU

M.G. Mustafa
SINP

M. Strickland
Kent State U.

N. Su
Bielefeld U.

Thank you for your kind attention.

Back Up: IR problem in pQCD

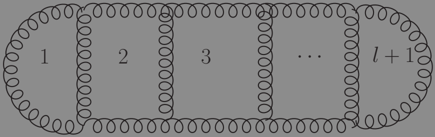

Figure: Divergent $(l+1)$-order loop diagrams

$$g^{2l}(T\int d^3k)^{l+1}k^{2l}(k^2+m^2)^{-3l}$$

- For $l=3$: $g^6T^4\ln\frac{T}{m}$
- For $l>3$: $g^6T^4\left(\frac{g^2T}{m}\right)^{l-3}$
- For $l\geq 3$: one needs to calculate infinite number of diagrams

Back Up: Charge Fluctuations

- If the expansion is fast, fluctuations of locally conserved quantities show a distinctly different behavior in a Hadron Gas (HG, unit of charge 1) and a quark-gluon plasma (QGP, unit of charge $\frac{1}{3}$).
- A suitable ratio is needed whose fluctuation is easy to measure and simply related to the charge fluctuation.
- Taking the ratio $F = \frac{Q}{N_{ch}}$ and considering a pion gas and a QGP consisting u and d quarks and gluons, it has been seen that the pion gas result is almost four times the QGP result.

  [Phys. Rev. Lett 85, 2076 (2000)]

Back Up: Overcounting

- At LO one obtains only the correct perturbative coefficients for the g^0 and g^3 terms when one expands in a strict power series in g.
- At NLO one obtains the correct g^0, g^2 and g^3 coefficients and at NNLO one obtains the correct coefficients upto $O(g^5)$.
- The resulting approximants obtained when going from LO to NLO to NNLO are expected to show improved convergence since the loop expansion is now explicitly expanded in terms of the relevant high-temperature degrees of freedom (quark and gluon high-temperature quasiparticles).