# Criticality and the equation of state of dense QCD

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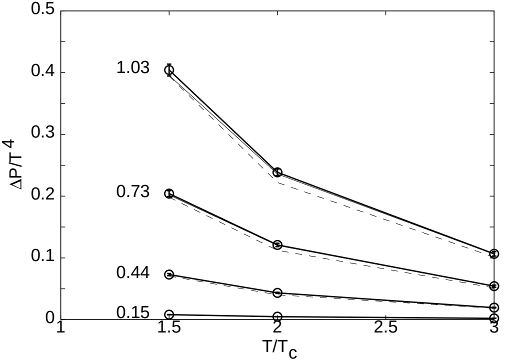
# Outline

# EOS at $\mu \neq 0$




**Gavai, SG: Phys.Rev. D68 (2003) 034506**

$$\Delta P = P(T,\mu) - P(T,0), \qquad n(T,\mu) = \frac{\partial P}{\partial \mu}.$$

**Recently— Borsanyi et al: JHEP 1208 (2012) 053; Hegde et al, 2014.**

# The mathematical problem

Given the series expansion of a function

$$\Delta P(T,\mu) = \sum_{i=0,2}^{\infty} \chi_i(T) \frac{\mu^i}{i!},$$

at $\mu = 0$, reconstruct the function. Well studied classical problem!

**Madhava, 14th century, or earlier in the same school**

Special complications: coefficients are known within errors; only a small number of coefficients are known.

- Simplest part of the problem: estimate whether the series is summable. Usual notions of radius of convergence.
- Next more complicated: estimate the value of the function defined by the series at $z \simeq \sqrt{f_0/f_2}$.

# Outline

# Simulation parameters

Older $N_f = 2$ rooted staggered simulations—

1. $N_t = 4$ with $m_\pi \simeq 220$ MeV
2. $N_t = 6$ with $m_\pi \simeq 270$ MeV
3. $N_t = 8$ with $m_\pi \simeq 310$ MeV

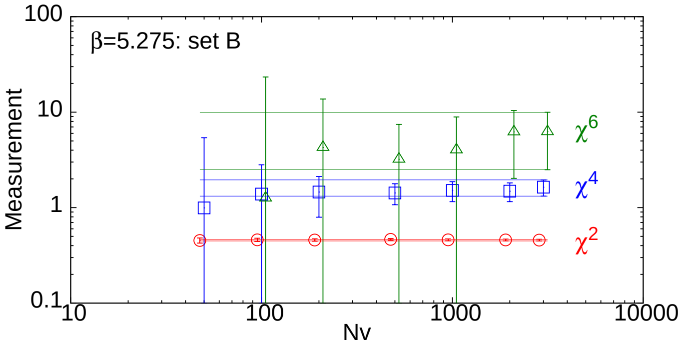
Spatial volumes of 100–300 $\mathrm{fm}^3$. **Datta, Gavai, SG: Lattice 2013**

New results on quark mass dependence on coarse lattices ($N_t = 4$)—

1. Set A: $m_\pi \simeq 450$ MeV
2. Set B: $m_\pi \simeq 220$ MeV
3. Set C: $m_\pi \simeq 190$ MeV

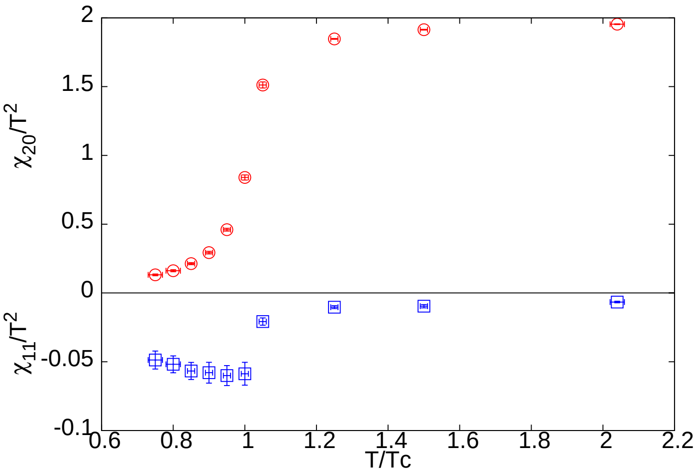
**SG, Karthik, Majumdar: 2014**

# Statistics




$N_s = 200$ independent gauge configs (40000 trajectories). $N_v$ is the number of source vectors used for noisy estimators of inverse fermion matrix.

# Susceptibilities at $\mu = 0$




**SG, Karthik, Majumdar: 2014**

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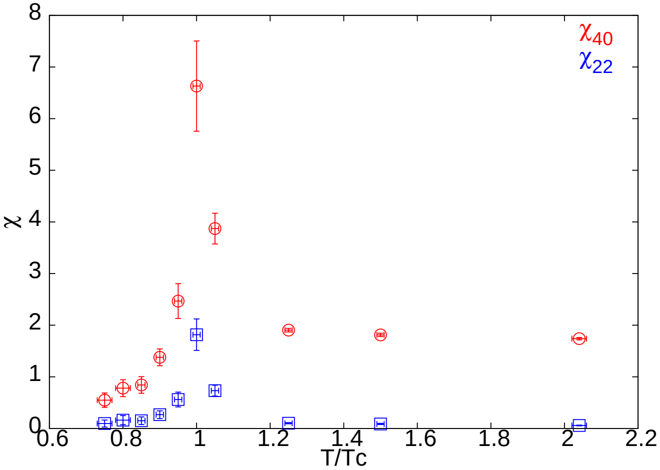


**SG, Karthik, Majumdar: 2014**

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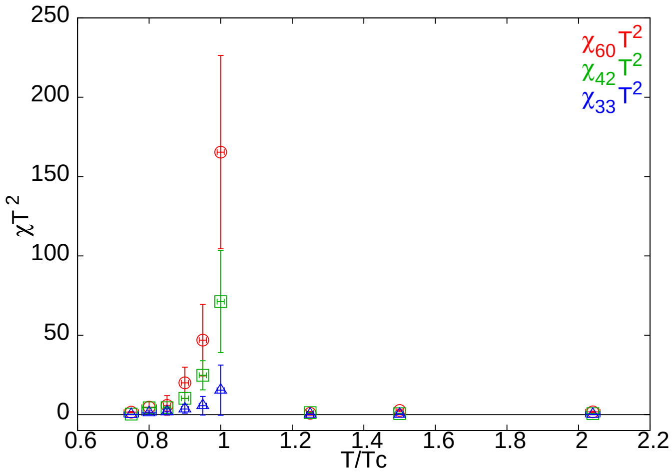
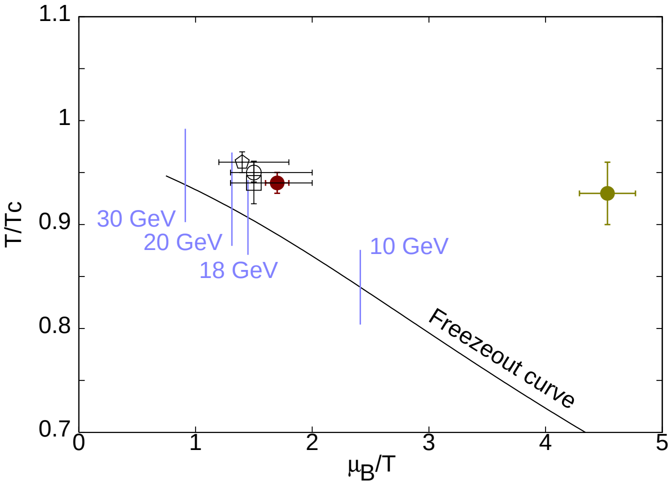



**SG, Karthik, Majumdar: 2014**

# Radius of convergence



# Outline

# Critical behaviour

The free energy of a system may be decomposed into a regular and a singular part,

$$F(T,m) = F_r(T,m) + F_s(T,m),$$

where the regular part, $F_r$, is smooth at the critical point. The singular part, $F_s$ has a scaling form

$$F_s(T,m) = t^{2-\alpha}\Phi(h), \quad \text{where } t = \left|1 - \frac{T}{T_c}\right|, \quad h = \frac{t}{m^{1/\beta\delta}}.$$

As a result,

$$c_V = \frac{\partial^2 F}{\partial T^2} \simeq t^{-\alpha},$$

and the specific heat diverges at $T_c$ with a critical exponent $\alpha$, as long as $\alpha > 0$.

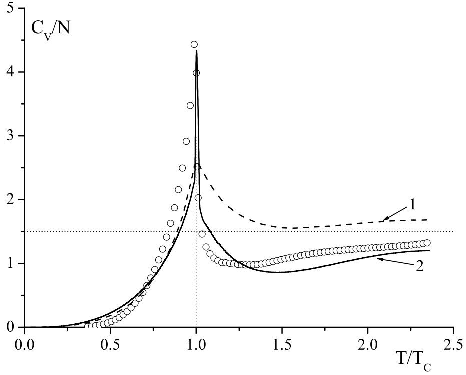
# O(N) critical exponents in 3D

| | | $\beta$ | $\delta$ | $\alpha$ | |
|---|---|---|---|---|---|
| O($\infty$) | | 1/2 | 5 | -1 | Antonenko et al, 1995 |
| O(4) | chiral QCD | 0.380 | 4.86 | -0.2268 | Engels et al, 2000 |
| O(3) | ? | 0.365 | 4.79 | -0.115 | Zinn-Justin et al, 1977 |
| O(2) | liquid He | 0.349 | 4.78 | -0.0172 | Engels et al, 2000 |
| O(1) | liquid-gas | 0.325 | 4.8 | 0.11 | Zinn-Justin et al, 1977 |
| MFT | | 1/2 | 3 | 0 | |

Specific heat exponent, $\alpha$, always negative for $N > 1$.

But, for liquid He, experiments show a peak in $c_V$ at $T_c$. Why?

# The $\lambda$ point of liquid Helium




**Vakarchuk, Pastukhov, Prytula, arxiv:1110.3941**

When $\alpha$ is negative, contribution of $F_s$ to $c_V(T_c) = 0$. So the value of $c_V(T_c)$ is entirely due to the regular part. There is a cusp at $T_c$: non-analyticity, must be due to $F_s$.

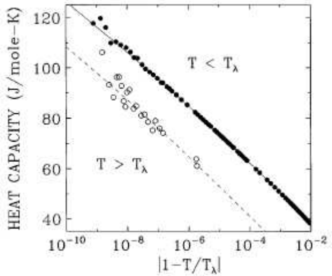
# Critical indices from the $\lambda$ point

When $|T - T_c|$ is small enough, contribution of regular part can be taken to be a constant. Then

$$c_V = A + t^{-\alpha}(B + Ct^{\Delta}),$$

where $\Delta$ is a possible sub-leading exponent. $A$ is constrained to be positive, so $B$ must be negative.

By taking explicit derivatives, it can be shown that it is possible to do this with $\Phi(h) > 0$. So the internal energy need not be negative.




**Lipa et al, PRL 76, 944 (1996)**

Space shuttle based experiment measured $c_p$ for $|T - T_c| \leq 2$ nK. Found $\alpha = -0.01285(38)$.

# The NJL model

The two flavour NJL model has the O(4) symmetry of QCD in the chiral limit. The quark mass $m$ is the analogue of the magnetic field. Then

$$c_V(T,m) = A + t^{-\alpha}\Psi(h), \quad \text{where } t = \left|1 - \frac{T}{T_c}\right|, \quad h = \frac{t}{m^{1/\beta\delta}}.$$

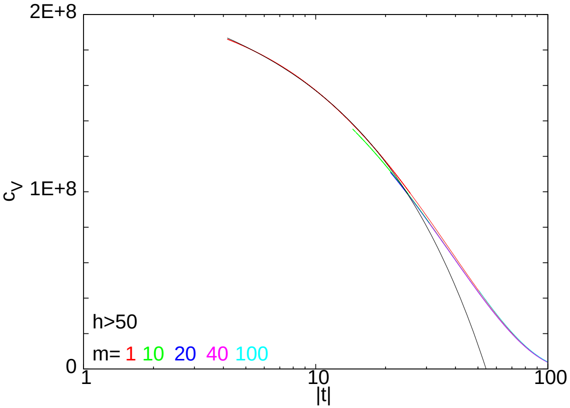
$T_c$ and other non-universal features change between NJL and PNJL, but universal critical features remain the same. In particular, $\Psi(h)$ can be computed in any model.

1. No singularities if $T \to T_c$ at fixed $m$, *i.e.*, $t \to 0$ and $h \to 0$.
2. $\lambda$ point can be seen only when $m \to 0$ before $T \to T_c$. Must take $h \to \infty$ first and then $t \to 0$.

Value of $c_V(T_c)$ in QCD cannot be predicted from a model.

**SG and Sharma, 2014**

# The specific heat




Data collapse successful when small $h$ is removed. What controls how large $h$ should be? Combinations of $F_\pi$ and $\langle\overline{\psi}\psi\rangle$.

## The quark number susceptibilities

CP symmetry implies symmetry $\mu \leftrightarrow -\mu$. As a result, the critical line is even in $\mu$—

$$T_c(\mu) = T_c + \frac{1}{2}\kappa\mu^2 + \cdots$$

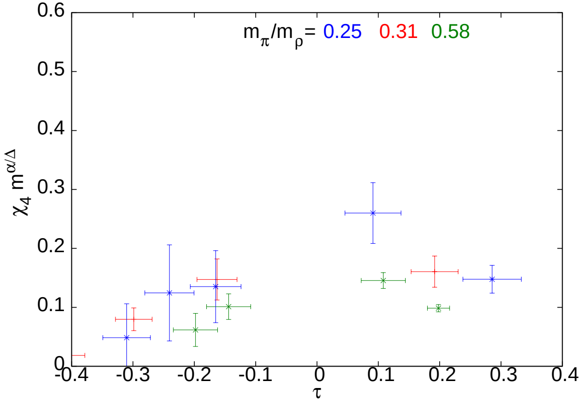
Then, for chiral QCD, if one assumes that the scaling function depends on $\mu$ only through $T_c(\mu)$, one finds

$$\frac{\partial}{\partial t} g(t,h)\bigg|_{\mu=0} = \frac{2T_c}{\kappa}\frac{\partial^2}{\partial\mu^2} g(t,h)\bigg|_{\mu=0}$$

when the derivatives are applied to scaling functions. As a result, $\chi_4 \propto c_V$ in the chiral limit.

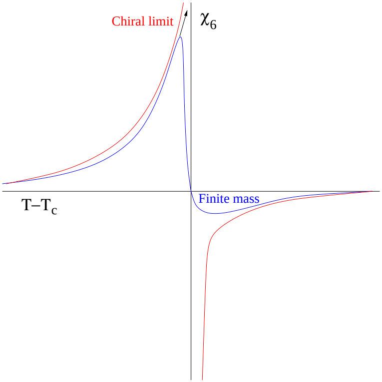
Caveat: The more complicated possibility of a new scaling variable $m(\mu)$, has not been ruled out. For now use the simplest possibility $m(\mu) = m$.

# Scaling via data collapse




$N_f = 2$ QCD with $N_t = 4$, and O(4) exponents. Largest pion mass may be outside the scaling regime. **SG, Sharma: 2014**

# $\chi_6$ is universal




Critical exponent: $\alpha_6 = 1 + \alpha$, so singular contribution dominates as $m = 0$. Scaling analysis by data collapse possible, when errors become smaller.

# Outline

# Samma: the right way

Truncating the sum

$$\Delta P(T,\mu) = \chi_2(T)\frac{\mu^2}{2} + \chi_4(T)\frac{\mu^4}{24}$$

is definitely wrong.

Need a method to compute $\Delta P$ which

1. keeps track of the radius of convergence of the series
2. gives an understanding of the way statistical errors in the measurements of the QNS affect the errors in $\Delta P$ and $n$
3. is transparently able to handle improved statistics and consequent increase in the number of measurable terms in the series expansion
4. is able to extract critical exponents from the measurements

Such a method is available: **ILGTI, Lattice 2013**

# The DLOG Pade

At a critical point $\chi_B \simeq (z_*^2 - z^2)^{-\psi}$. Since

$$\chi_B = \frac{\partial^2 (P/T^4)}{\partial z^2},$$

the continuity and finiteness of $P$ at the CEP forces $\psi \leq 1$.

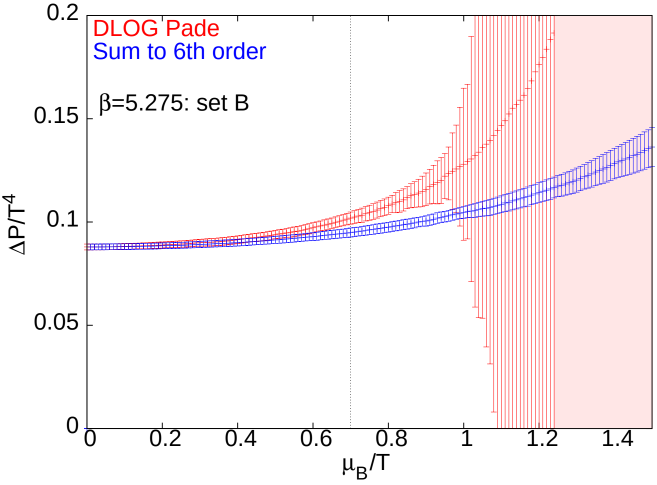
Branch cut: hard to do a simple Padé analysis. Well-known technique, convert to a problem with a pole:

$$m_1(z) = \frac{d \log \chi_B}{dz} \simeq \frac{2\psi z}{z_*^2 - z^2}.$$

Use the series to estimate the critical exponent.

Integrate the Padé to get $\chi_B$, then integrate twice more to get successively $n$ and $\Delta P$.

# Is the resummation useful?



# $m_1$ for $\mu \neq 0$

Resum the series into a Padé approximant—

$$[0,1]: \qquad f(z) = \frac{m_1(z)}{z} = \frac{c}{z_* - z}$$

Width of the critical region? If we define it by

$$\left| \frac{f(z)}{f(0)} \right| > \Lambda,$$

then this implies $|z - z_*| \leq z_*/(c\Lambda)$.
Errors in extrapolation?

$$\left| \frac{\Delta f}{f} \right| > \frac{1}{1 - \Lambda\delta},$$

where $\delta$ is fractional error in $z_*$.

Even if $\delta$ is controlled with huge statistics, close enough to the critical point $\Lambda$ is large; the error in $m_1$ becomes large!

# The “no free lunch” theorem

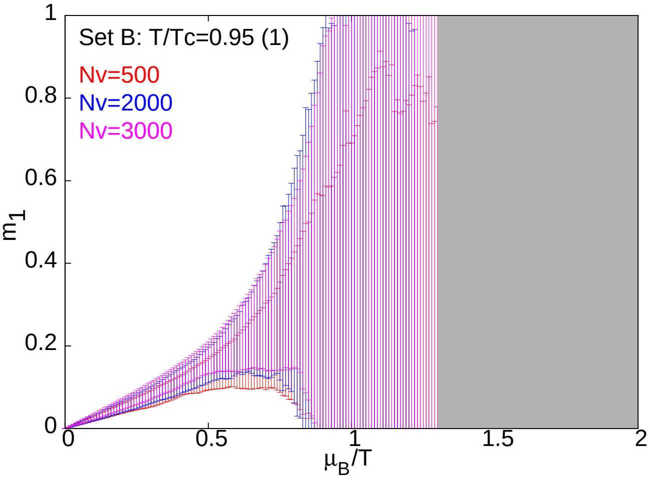
## Critical point has large fluctuations

Sampling these fluctuation requires very large CPU time: critical slowing down.

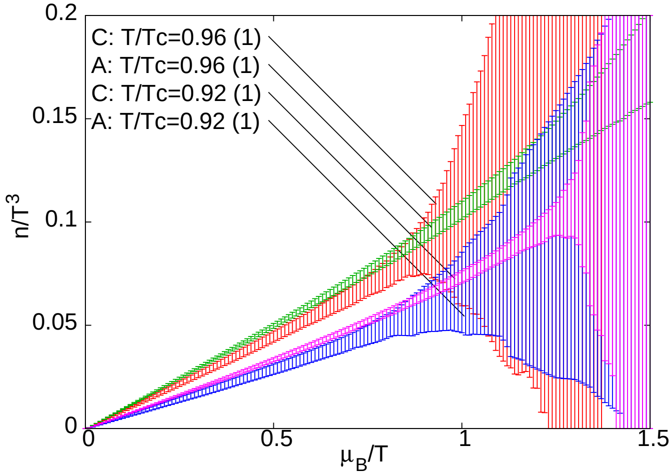
1. in direct Monte Carlo simulations: long autocorrelations.
2. in numerical series expansions: large statistical errors.

Large statistics needed in order to measure physical quantities near a critical point. In usual Monte Carlo simulations: large autocorrelation lengths signal approach to a critical point. In numerical series expansions: strong statistics dependence of extrapolations.

# Critical slowing down



# Quark mass dependence of equation of state



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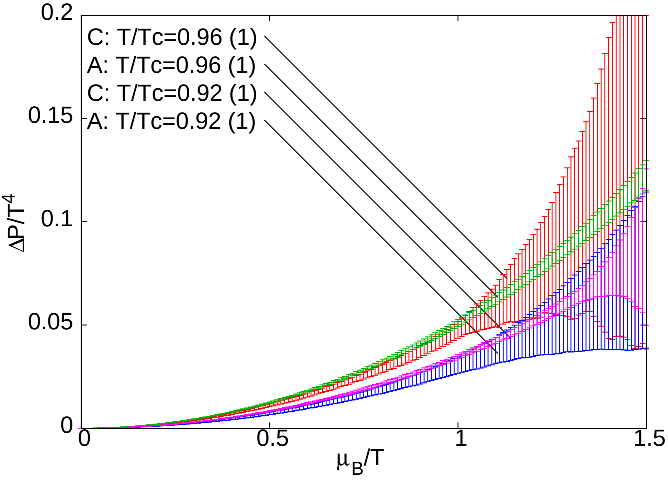

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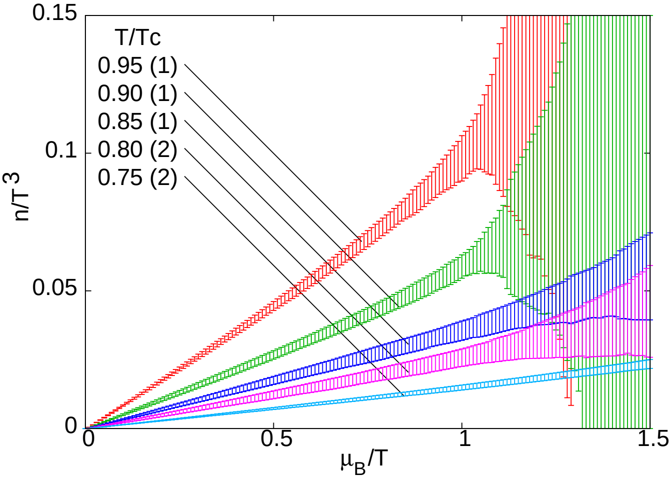

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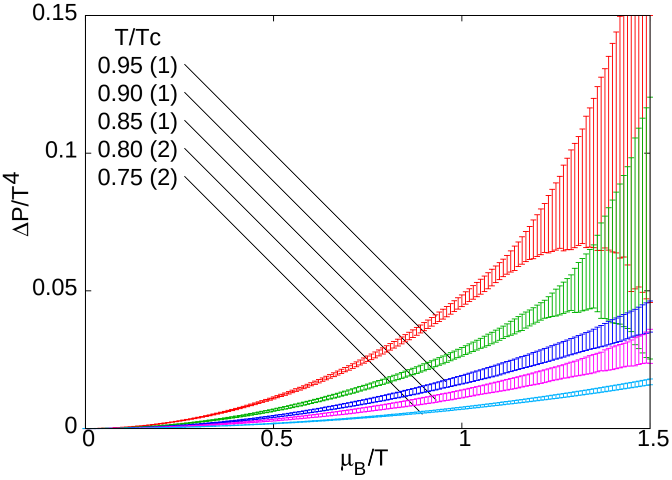

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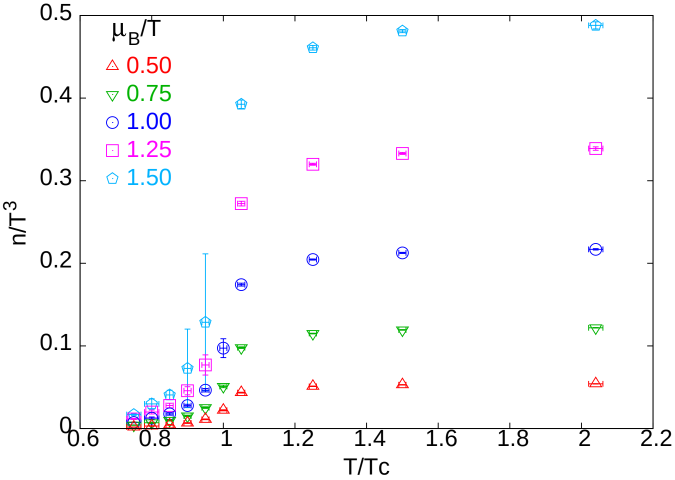

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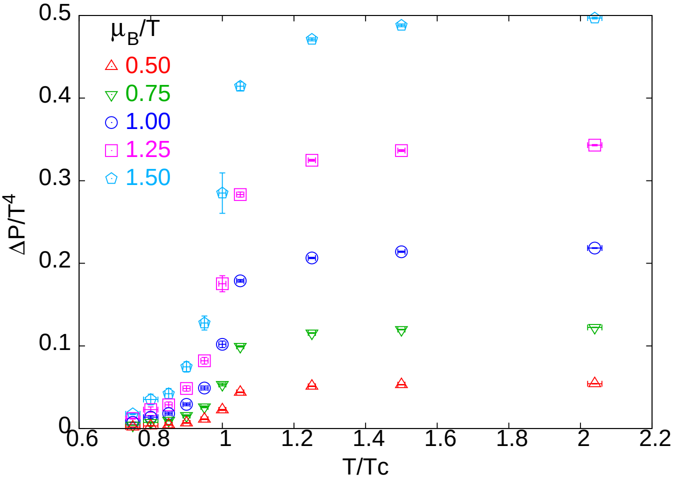

# Outline

# Equation of State

1. Measurement of quark number susceptibilities is source hungry. (Algorithmic improvements highly welcome). Currently measure 3–4 terms. Little quark mass dependence of the radius of convergence.
2. Naive O(4) scaling prediction of data collapse for QNS measurements with different quark masses. Reasonably well under control, requires much larger statistics to improve.
3. DLOG Padé analysis of QNS indicates radius of convergence compatible with an Ising critical point. Critical slowing down visible in statistical error propagation. Equation of state extracted. Measurement of critical exponent $\psi$ possible.
4. Method under control: still early days for statistical control.

# The real statistical picture