Induced QCD with $N_c - 1$ auxiliary bosonic fields

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1. Motivation

Limitations of LQCD – Why changing the gauge action?

Main problem for studies of the QCD phase diagram:

- **Simulating QCD at (real) non-zero chemical potential.** (sign problem)

Possible solutions:

- Use complex Langevin for simulations.
  [Parisi, PLB 131 (1983); Aarts, Stamatescu, JHEP 0809 (2008); Sexty, PoS LAT (2014)]
- Simulate on a Lefschetz thimble? [Christoforetti *et al*, PRD 86 (2012); PRD 88 (2013)]
- Dual variables and worm algorithms
  [e.g. Delgado Mercado *et al*, PRL 111 (2013), Gattringer, Lattice 2013]
- Fermion bags [e.g. Chandrasekharan, EPJA 49 (2013)]

Often it is the gauge action which makes it difficult to find solutions.
(see e.g. strong coupling solution to sign problem [Karsch, Mütter, NPB 313 (1989)])

Idea: Find an alternative discretisation of pure gauge theory which allows the use of strong coupling methods!

⇒ A gauge action which is linear in the gauge fields might do this job!

Induced QCD

This idea is not new!

Ansatz: Induce pure gauge dynamics using auxiliary fields.

- Using fermionic fields:
  - with standard (Wilson) fermions. [Hamber, PLB 126 (1983)]
  - Standard fermions + 4-fermion current-current interaction. [Hasenfratz, Hasenfratz, PLB 297 (1992)]

  Need the limit $N_f \to \infty$, $\kappa \to 0$.
- Using scalar fields:
  - Spin model. [Bander, PLB 126 (1983)]
    Needs the limit $N_s \to \infty$ and $g_s \to \infty$.
  - Adjoint scalar fields. [Kazakov, Migdal, NPB 397 (1992)]
    No "exact" pure gauge limit.
    It is interesting since it allows a solution in terms of large N_C.

$\Rightarrow$ This is where our induced model offers improvement!

Lattice regularised path integrals – fixing notations

Expectation value of operator O:

$$\langle O \rangle = \frac{1}{Z} \int [dU][d\psi][d\bar{\psi}]\; O\, \omega_G[U]\, \omega_F[\psi, \bar{\psi}, U]$$

- $\omega_G[U]$: Pure gauge weight factor.
- $\omega_F[\psi, \bar{\psi}, U]$: Quark weight factor.

Typically: $\omega_G[U]\, \omega_F[\psi, \bar{\psi}, U] = \exp\left[-S[\psi, \bar{\psi}, U]\right]$.

Basic demands:

- The discretised action should preserve the continuum symmetries.
- The discretised action has to reproduce the continuum Yang-Mills action.

2. The new weight factor

Zirnbauer's weight factor

Consider the weight factor: [Budczies, Zirnbauer, math-ph/0305058]

$$\omega_{\mathrm{BZ}}[U] \sim \prod_p \left| \det \left(m_{\mathrm{BZ}}^4 - U_p \right) \right|^{-2N_b}$$

Here:

- p is an index running over unoriented plaquettes U_p.
- m_{BZ} is a real parameter with $m_{\mathrm{BZ}} \geq 1$
  (or more generally $m_{\mathrm{BZ}} \in \mathbb{C}$ with $\mathrm{Re}(m_{\mathrm{BZ}}) \geq 1$)
- N_b is a (integer) number
- we consider a hypercubic lattice

Does this weight factor have anything to do with continuum Yang-Mills theory?

Why is this weight factor interesting?

The naive pure gauge limit

There is one obvious way to establish a connection:

- Write the weight factor as:

$$\omega_{\mathrm{BZ}}(U) \sim \exp\left\{-2\ N_b\ \mathrm{Re}\left[\sum_p \mathrm{Tr}\ln\left(1-\alpha_{\mathrm{BZ}}\ U_p\right)\right]\right\}$$

with $\alpha_{\mathrm{BZ}} = m_{\mathrm{BZ}}^{-4}$

- Expand the exponent in small α_{BZ}:

$$\Rightarrow \quad S_{\mathrm{BZ}}^{\mathrm{eff}}(U) = -2\ N_b \sum_p \left[\alpha_{\mathrm{BZ}}\ \mathrm{ReTr}\left(U_p\right) + \mathcal{O}(\alpha_{\mathrm{BZ}}^2)\right]$$

- Comparison with the Wilson action:

Equivalent if $\beta = N_b\ N_c\ \alpha_{\mathrm{BZ}}$!

$\Rightarrow$ Pure gauge limit: $\alpha_{\mathrm{BZ}} \to 0 \quad N_b \to \infty \quad$ (so that β fixed)

Non-trivial pure gauge limit for U(N_c)

Zirnbauers conjecture: [Budczies, Zirnbauer, math-ph/0305058]

At fixed $N_b \geq N_c$ and $d \geq 2$ the theory for gauge group U(N_c) has a continuum limit for $\alpha_{\mathrm{BZ}} \to 1$ which reproduces continuum Yang-Mills theory.
(excluding the case $d = 2$ and $N_b = N_c$)

In [math-ph/0305058] they give a proof for:

- The existence of a continuum limit for all numbers of dimension and $N_b \geq N_c$.
- The equivalence with Yang-Mills theory for $d = 2$ and $N_b > N_c$.

For $d > 2$ the equivalence with Yang-Mills theory is only a conjecture and relies on the increase of the collective behaviour of the gauge field when going to $d > 2$.

Conjecture: This should also work for other gauge groups.

Non-trivial pure gauge limit for SU(N_c)

The extension of the proof to SU(N_c) is not completely straightforward!
(Exception: SU(2) case)

We have redone the proof for the existence of a continuum limit using a method which differs from the one of the U(N_c) case.

Result:

- The continuum limit exists for $N_b \geq N_c - \frac{5}{4}$.
  (The exact border for U(N_c) is $N_b \geq N_c - \frac{1}{2}$)
- It reproduces continuum Yang-Mills theory for $d = 2$ and $N_b > N_c - \frac{3}{4}$.
  (For U(N_c) the border is $N_b > N_c + \frac{1}{2}$)

For SU(N_c) the argument for the increase of collectivity of the gauge field when going to $d > 2$ still holds!

Excursus: Why not lattice perturbation theory?

Usually one would investigate the continuum limit via lattice perturbation theory.

Here: effective action

$$S^{\text{eff}} = -2\, N_b \,\text{Re}\left[\sum_p \text{Tr}\ln\left(1 - \alpha_{\text{BZ}}\; U_p\right)\right]$$

Expanding with $U = \exp(iga\,A)$ around $U_p = 1$ for $\alpha_{\text{BZ}} \to 1$:

$\Rightarrow$ Convergence radius vanishes for expansion around $1 - \alpha_{\text{BZ}}$.

Alternative possibility: Use large N_b perturbation theory.

- Systematic expansion possible only for $\alpha_{\text{BZ}} \lesssim 0.172$.
- Implies a suitable (Wilson like) coupling to be: $g^2 = \dfrac{2(1-\alpha_{\text{BZ}})}{N_b \alpha_{\text{BZ}}}$
- We can now compute the relation between the two couplings.

Problem: Does not apply directly to the continuum limit!

Non-trivial pure gauge limit for SU(N_c) – Numerical test

$\Rightarrow$ The only option is to probe the continuum limit numerically.

To check the existence of a continuum limit:

Consider the expectation value

$$\langle F \rangle_{\alpha_{\rm BZ}} = \frac{1}{Z} \int dU\, F(U)\, \left|\det\left(1 - \alpha_{\rm BZ} U\right)\right|^{-2N_b}$$

for some testfunction $F(U)$.

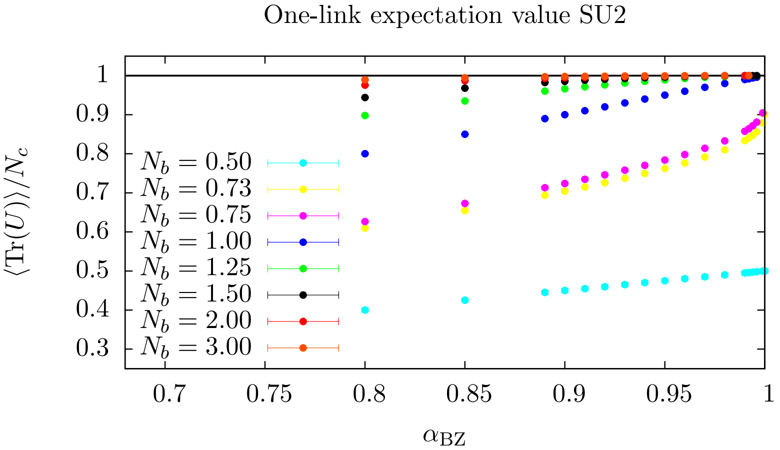
For $\alpha_{\rm BZ} \to 1$ we should have

$$\left|\det\left(1 - \alpha_{\rm BZ} U\right)\right|^{-2N_b} \to \delta^{\mathrm{SU}(N_c)}(U - 1)\,.$$

$\Rightarrow$ I.e. $\langle F \rangle_{\alpha_{\rm BZ}} \to F(1)$

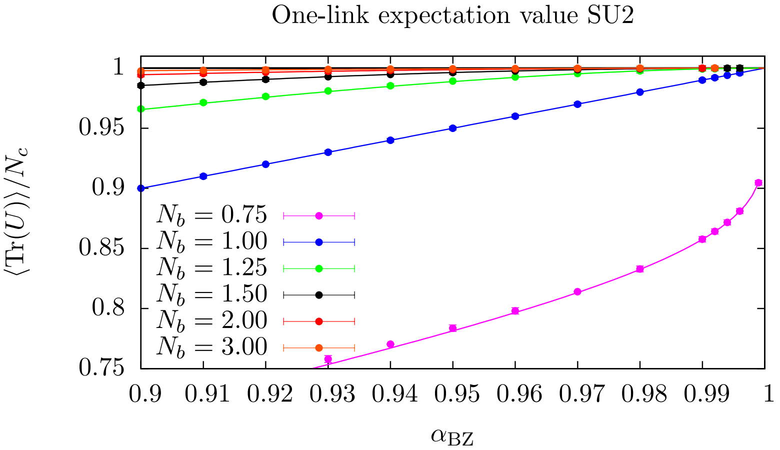
Non-trivial pure gauge limit for SU(N_c) – Numerical test

First: Lets look at $F(U) = \text{Tr}(U)/N_c$

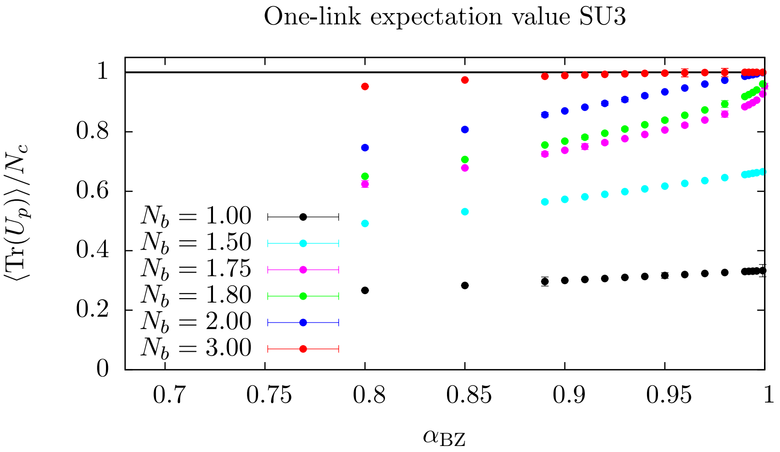


Non-trivial pure gauge limit for SU(N_c) – Numerical test

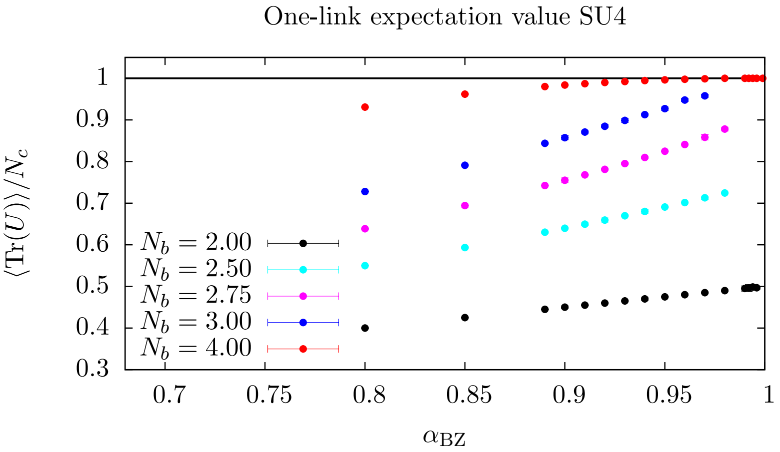
Analytical results for small $(\alpha_{\mathrm{BZ}} - 1)$ and SU(2) from character expansion:



Non-trivial pure gauge limit for SU(N_c) – Numerical test

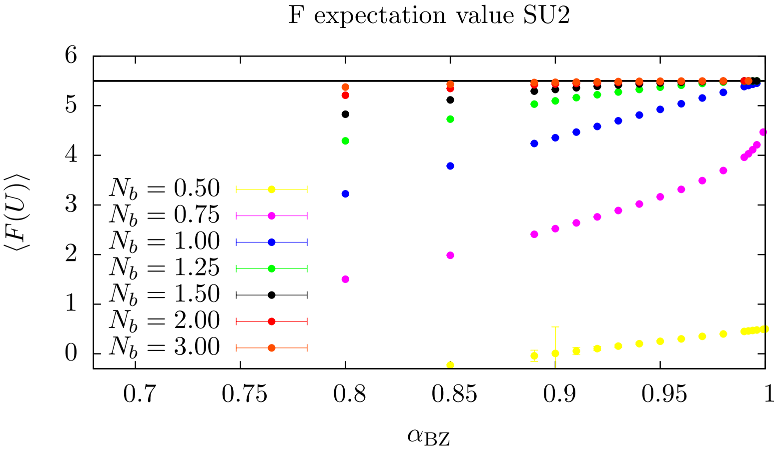


Non-trivial pure gauge limit for SU(N_c) – Numerical test

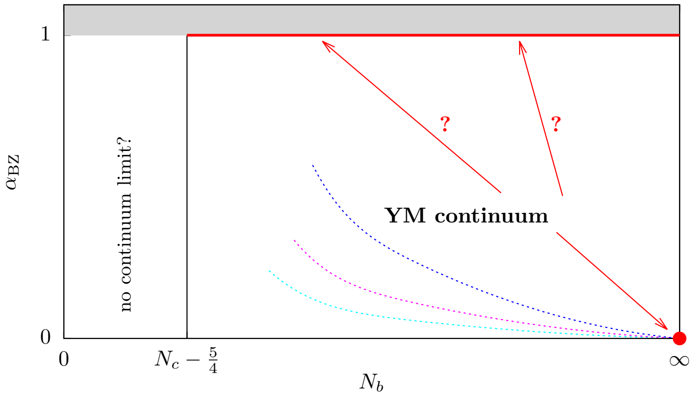


Non-trivial pure gauge limit for SU(N_c) – Numerical test

What about other test functions $F(U)$? (here $F(U) = \mathrm{Tr}[U + 6U^2 - 1.5U^3]$)



Phases in the $(N_b, \alpha_{\mathrm{BZ}})$ parameter space



$\Rightarrow$ We will now test the properties of the theory in this limit!

3. Numerical tests for 3d SU(2)

Basic idea and setup

First: Consider the cheap case SU(2) at $d = 3$!

Suitable observables for a first test:

- $T = 0$ observables:
  Quantities connected with the $q\bar{q}$ potential.
- $T \neq 0$ observables:
  Transition temperature and order of the transition.

Simulation setup:

- Wilson theory: Standard mixture of heatbath and overrelaxation updates.
- Induced theory: Local metropolis with links evolving in ϵ-ball.
- Computation of $q\bar{q}$ potential: Lüscher-Weisz algorithm

  [Lüscher, Weisz, JHEP 0109 (2010)]
- Scale setting: Sommer parameter r_0

  [Sommer, NPB 411 (1994)]

First step: Matching between α and β

- Start with some information from $\langle U_p \rangle$.
- Compute r_0 in the interesting region:

$\Rightarrow$ Matching via r_0/a:

$$\beta(\alpha) = \frac{b_{-1}}{1-\alpha} + b_0 + b_1(1-\alpha)$$

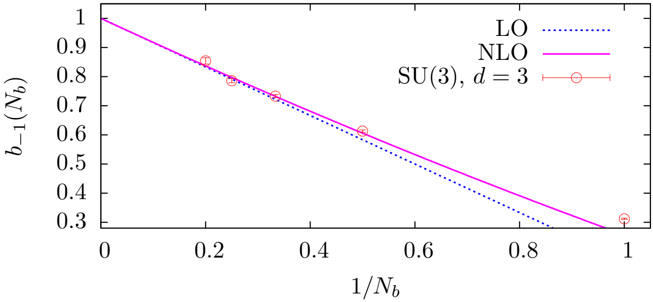
(consistent with perturbation theory)

N_b	b_{-1}	b_0	b_1
1	0.623(4)	-1.78(11)	3.59(69)
2	2.453(14)	-2.76(38)	0.99(5)
3	4.399(29)	-4.43(16)	-0.17(21)
4	6.286(52)	-6.01(23)	-0.52(25)
5	8.54 (11)	-8.99(41)	0.45(38)

Comparison to large N_b perturbation theory

b_{-1} in large N_b perturbation theory ($d = 3$, $N_c = 2$):

$$\frac{b_{-1}(N_b)}{N_c N_b} = 1 - \frac{5}{6N_b} + \frac{0.0908283}{N_b^2} + O(N_b^{-3})$$



Second step: Look at static potential at similar lattice spacings

- Compare to high precision results obtained with the Wilson action.

[BB, PoS EPS-HEP (2013)]

- At large distances R the energy levels of the $q\bar{q}$ boundstate are well described by an effective string theory!

[Nambu, PLB 80, 372 (1979); Lüscher, Symanzik, Weisz, NPB 173, 365 (1980), Polyakov, NPB 164, 171 (1980)]

Potential in effective string theory for the flux tube ($d = 3$):

[Aharony *et al*, JHEP 0906 (2009); JHEP 1012 (2010); JHEP 1101 (2011); JHEP 1305 (2013)]

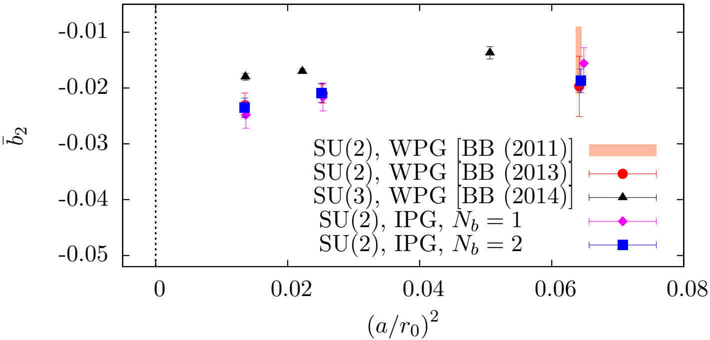
$$V(R) = \sigma R \sqrt{1 - \frac{\pi}{12\sigma R^2}} - \bar{b}_2 \frac{\pi^3}{60\sqrt{\sigma^3} R^4}$$

$\Rightarrow$ There are two non-universal parameters, σ and $\bar{b}_2$ (boundary coeff.).

- First result: $\sqrt{\sigma}\, r_0$ is equivalent in both theories!
- An agreement of $\bar{b}_2$ means that the potential is identical up to 4-5 significant digits!

Results for $\bar{b}_2$

Results for $\bar{b}_2$:



⇒ All results are in excellent agreement!

Finite T properties

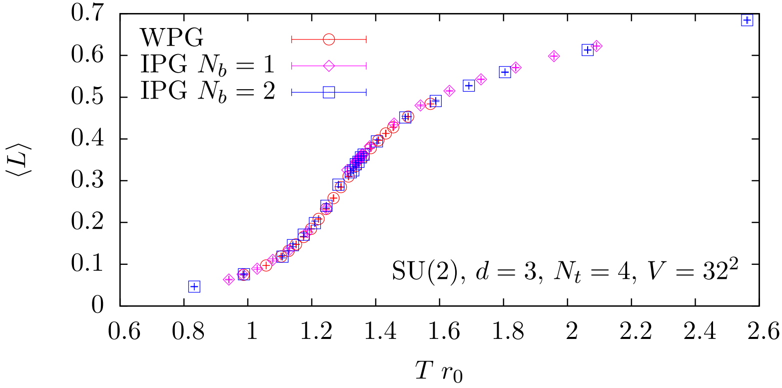
For $T = 0$ quantities comparison looks good!

So what about the finite temperature transition?

- For SU(2) and $d = 3$:
  Second order phase transition in the $2d$ Ising universality class.
  [Engels *et al*, NPPS 53 (1997)]
- We will test this at $N_t = 4$ first!
  $\Rightarrow$ For $N_t = 6$ I present some first results.
- Scale setting via r_0 and the mapping obtained at $T = 0$.

Phase transition at $N_t = 4$

Polyakov loop expectation value:



Phase transition at $N_t = 4$

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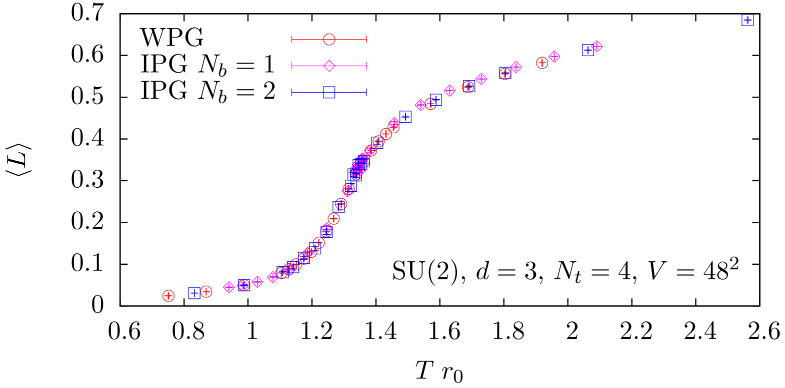

Phase transition at $N_t = 4$

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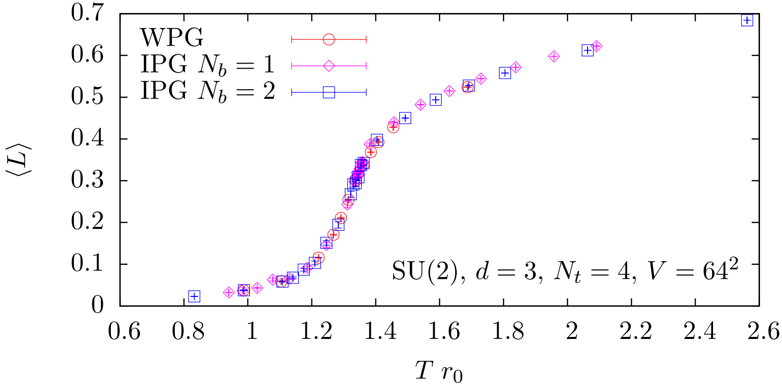

Phase transition at $N_t = 4$

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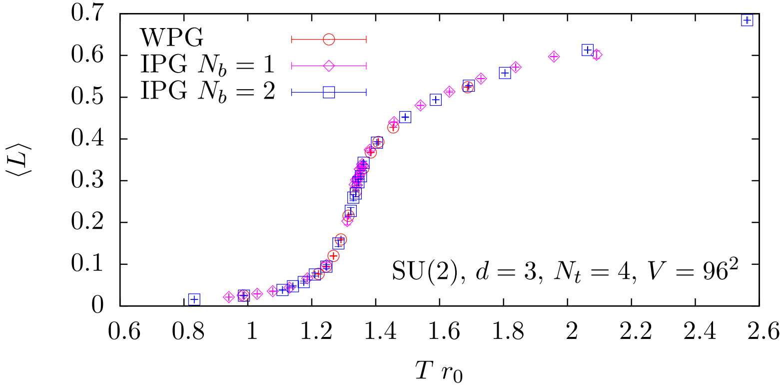
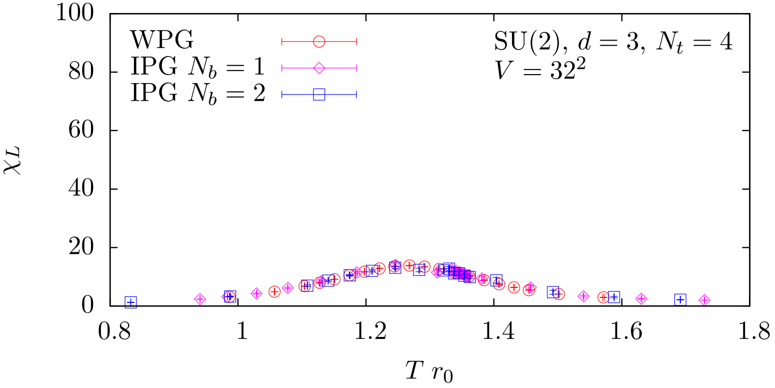


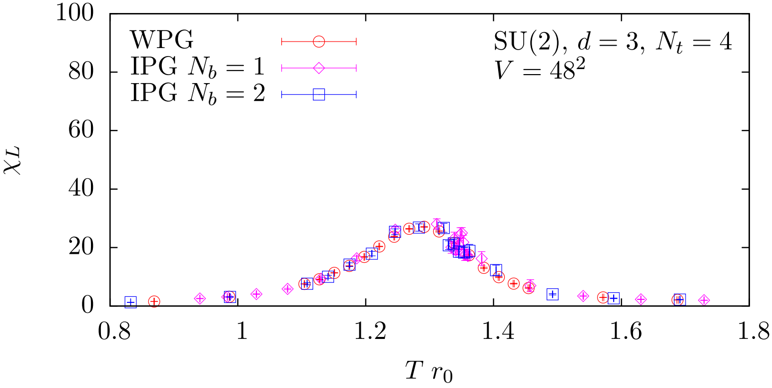
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



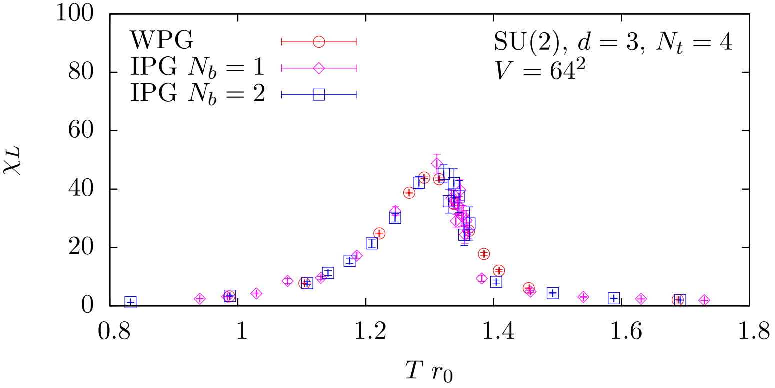
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



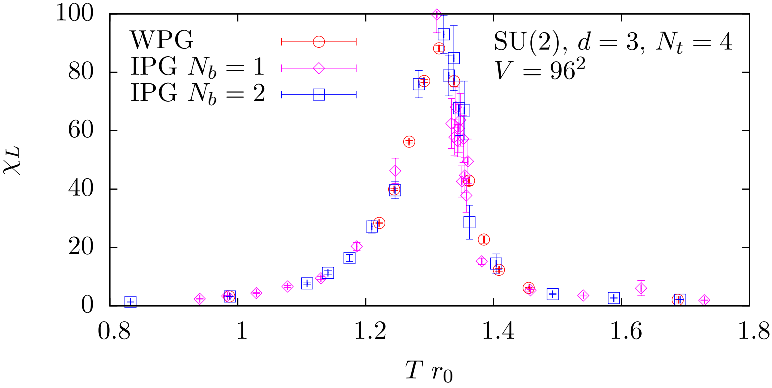
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



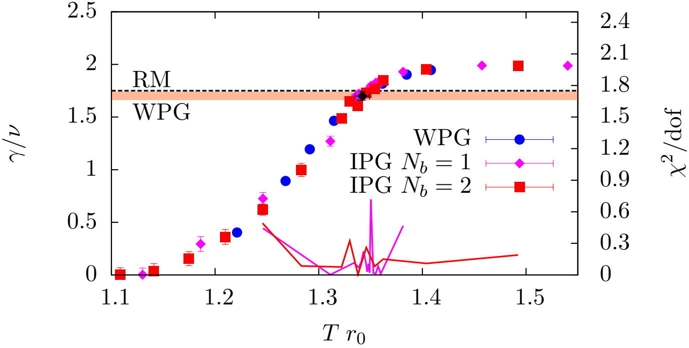
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



Phase transition at $N_t = 4$

Fit: $\ln(\chi_L) = C + \gamma/\nu \ln(N_s)$

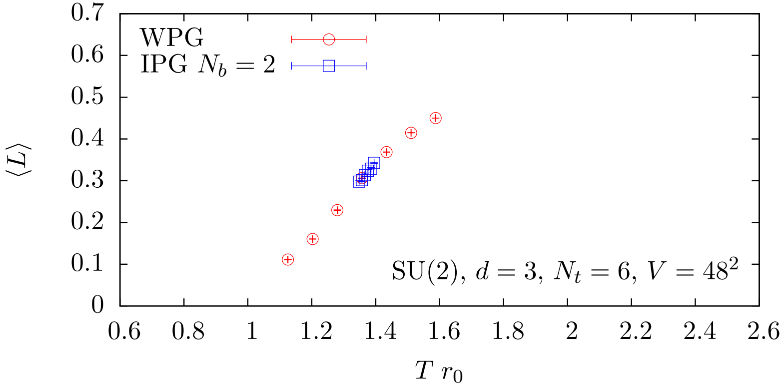


Black point: $\gamma/\nu = 1.70(4)$ (WPG)

[Engels *et al*, NPPS 53 (1997)]

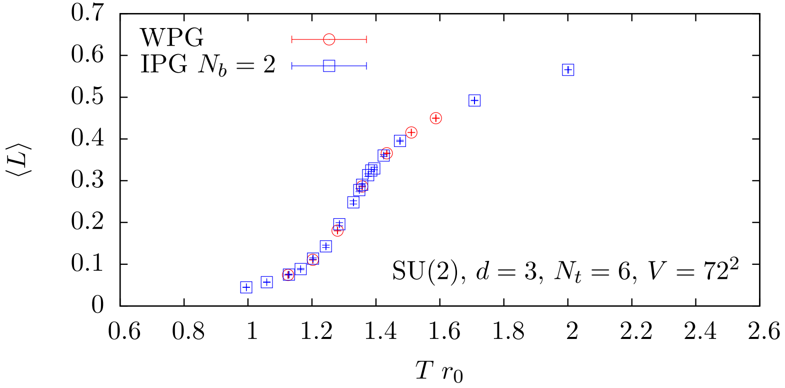
Phase transition at $N_t = 6$

Polyakov loop expectation value:



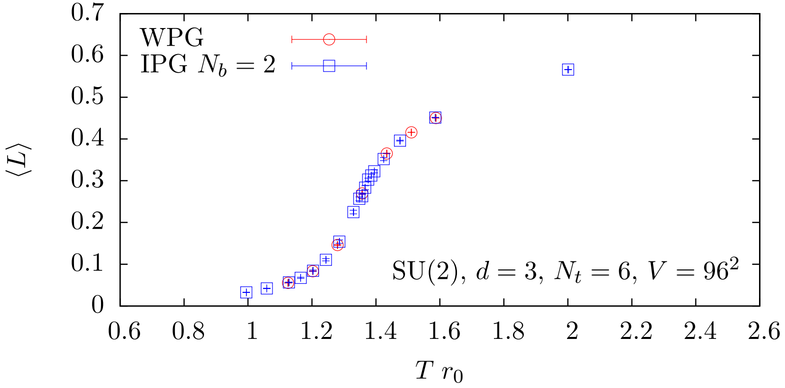
Phase transition at $N_t = 6$

Polyakov loop expectation value:



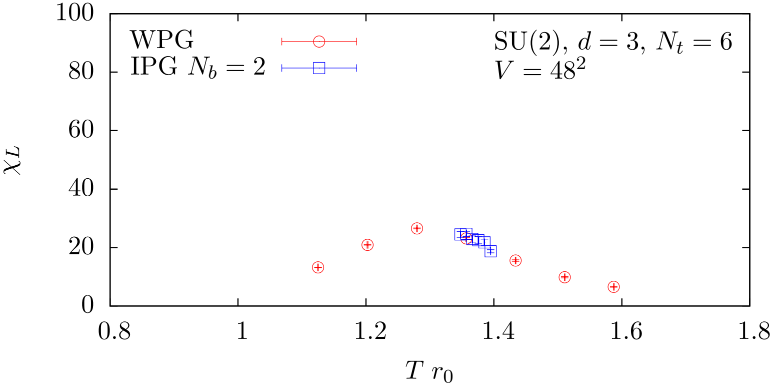
Phase transition at $N_t = 6$

Polyakov loop expectation value:



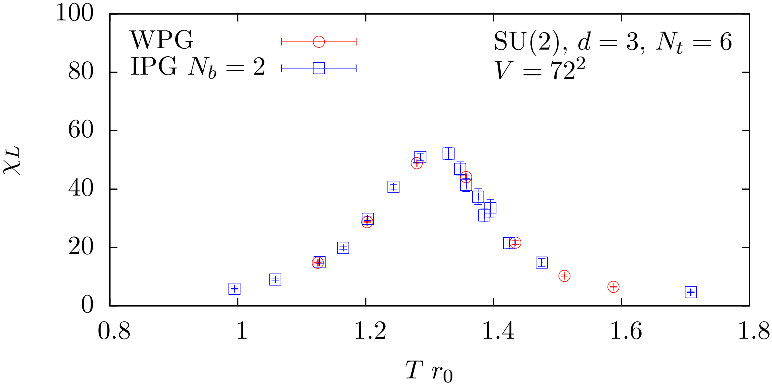
Phase transition at $N_t = 6$

Polyakov loop susceptibility:



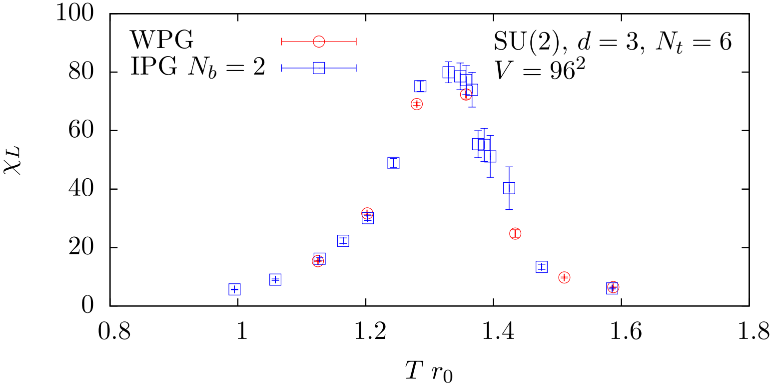
Phase transition at $N_t = 6$

Polyakov loop susceptibility:



Phase transition at $N_t = 6$

Polyakov loop susceptibility:



4. Dual representation

The bosonic version

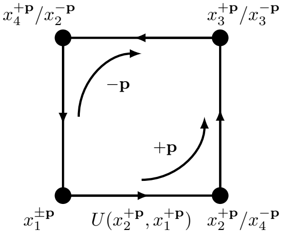
Now: Why is this weight factor interesting?

Bosonisation of the determinant: [Budczies, Zirnbauer, math-ph/0305058]

$$\omega_{\mathrm{BZ}}[U] = \prod_p \left|\det\left(m_{\mathrm{BZ}}^4 - U_p\right)\right|^{-2N_b} = \int [d\bar{\phi}][d\phi]\ \exp\left\{-S_{\mathrm{BZ}}[\phi,\bar{\phi},U]\right\}$$

$$S_{\mathrm{BZ}}[\phi,\bar{\phi},U] = \sum_{b=1}^{N_b}\sum_{\pm\mathbf{p}}\sum_{j=1}^{4}\left[m_{\mathrm{BZ}}\,\bar{\phi}_{b,\mathbf{p}}(x_j^{\mathbf{p}})\phi_{b,\mathbf{p}}(x_j^{\mathbf{p}}) - \bar{\phi}_{b,\mathbf{p}}(x_{j+1}^{\mathbf{p}})\,U(x_{j+1}^{\mathbf{p}},x_j^{\mathbf{p}})\,\phi_{b,\mathbf{p}}(x_j^{\mathbf{p}})\right]$$

- ϕ are complex scalar fields
- **p**: index for oriented plaquette
- Scalar fields carry plaquette index **p**.
  ⇒ Propagate only locally opposite to the plaquette orientation.
- Gauge field only couples to bosons.
  ⇒ Can be modified more easily!
- N_b defines the number of boson fields.



Modified version

Problem: This action is complex!

Solution: Rewrite determinant weight factor:

$$\omega_{\mathrm{BZ}}[U] \sim \prod_p \left[\det\left(m_{\mathrm{BZ}}^4 - U_p\right) \det\left(m_{\mathrm{BZ}}^4 - U_p^\dagger\right)\right]^{-N_b}$$

$$\sim \prod_p \left[\det\left(\tilde{m} - \left\{U_p + U_p^\dagger\right\}\right)\right]^{-N_b}$$

Now bosonize this determinant:

⇒ Real action:

$$S_B[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} \left[m\, \bar{\phi}_{b,p}(x_j)\phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_{j+1})\, U(x_{j+1}, x_j)\, \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_j)\, U(x_j, x_{j+1})\, \phi_{b,p}(x_{j+1})\right]$$

Here: $\tilde{m} = m_{\mathrm{BZ}}^4 + m_{\mathrm{BZ}}^{-4}$ and $\tilde{m} = m^4 - 4\, m^2 + 2$.

Integration over gauge fields

Rewrite the partition function as a product of integrals:

$$\begin{aligned} Z &= \int [d\bar{\phi}][d\phi]\, \mathcal{F}[\phi,\bar{\phi}] \prod_{x,\mu} \int dU_\mu(x)\, e^{\frac{1}{2}\mathrm{Tr}\left[U_\mu(x)\; A_\mu(x)[\phi,\bar{\phi}] + U^\dagger_\mu(x)\; A^\dagger_\mu(x)[\phi,\bar{\phi}]\right]} \\ &= \int [d\bar{\phi}][d\phi]\, \mathcal{F}[\phi,\bar{\phi}] \prod_{x,\mu} \mathcal{I}_{x,\mu}[\phi,\bar{\phi}] \end{aligned}$$

With $\mathcal{F}[\phi,\bar{\phi}] = \exp\left\{ -\sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} m\, \bar{\phi}_{b,p}(x_j)\phi_{b,p}(x_j) \right\}$

and $$A_\mu(x)[\phi,\bar{\phi}] = 2\sum_{b=1}^{N_b}\sum_{\nu\neq\mu} \left[\phi_{b,\bar{p}(x,\mu,\nu)}(x_{\bar{j}(\mu,\nu,0,1)})\bar{\phi}_{b,\bar{p}(x,\mu,\nu)}(x_{\bar{j}(\mu,\nu,0,0)}) + \phi_{b,\bar{p}(x-\hat{\nu},\mu,\nu)}(x_{\bar{j}(\mu,\nu,1,1)})\bar{\phi}_{b,\bar{p}(x-\hat{\nu},\mu,\nu)}(x_{\bar{j}(\mu,\nu,1,0)}) \right]$$

Integration over gauge fields

Need to solve integrals $\mathcal{I} = \int dU \, e^{\mathrm{Tr}\left[U\,A + U^\dagger\,A^\dagger\right]}$.

For U(N_c) they are known. [e.g. Brower, Rossi, Tan, PRD23 (1981)]

For SU(N_c): $\Rightarrow$ $$\mathcal{I} \sim \frac{1}{\Delta(\lambda^2)} \sum_{\xi=0}^{\infty} \varepsilon_\xi \, \cos(\xi\,\varphi) \, \det\left(K_\xi(\lambda)\right)$$

- ε_ξ: Neumann's factor; $\varepsilon_\xi = \begin{cases} 1 & \text{for}\xi = 0 \\ 2 & \text{for}\xi > 0 \end{cases}$
- φ: Phase of the determinant det(A)
- λ_i^2: eigenvalues of the $N_c \times N_c$ matrix $AA^\dagger$
- $\Delta(\lambda^2)$: Vandermonde determinant
- $K_\xi(\lambda)$: $N_c \times N_c$ matrix; $(K_\xi(\lambda))_{ij} = \lambda_i^{j-1}\, I_{\xi+j-1}(\lambda_i)$
  with $I_m(z)$ modified Bessel function of the first kind (and $z \in \mathbb{R}$).

$\Rightarrow$ Looks difficult, but the sum in $\mathcal{I}$ converges numerically very fast.

Full QCD

Now consider also fermionic fields, e.g. with a staggered type action:

$$S_F = \sum_x \Big\{ \sum_\mu \Big[\bar\psi(x)\alpha_\mu(x)\, U_\mu(x)\psi(x+\hat\mu) + \bar\psi(x+\hat\mu)\tilde\alpha_\mu(x)\, U_\mu^\dagger(x)\psi(x) \Big] + m_q \bar\psi(x)\psi(x) \Big\}$$

Expanding the weight factor, integrating over the grassmann variables and gauge field (following [Karsch, Mütter, NPB 313 (1989)]):

$$Z = \sum_{\{n,k,l_b,l_q\}} \Big\{ \prod_x \omega_x \prod_b \omega_b \prod_{l_b} \omega_{l_b} \Big\} \int [d\bar\phi][d\phi] \prod_{l_q} \{\omega_{l_q}[\phi,\bar\phi]\} \mathcal{F}[\phi,\bar\phi] \prod_b \mathcal{I}_b[\phi,\bar\phi]$$

- Monomer terms: $\omega_x = \frac{N_c!}{n_x!}(2am_q)^{n_x}$ with $n_x \in \{0,\ldots,N_c\}$
- Dimer terms: $\omega_b = \frac{(N_c-k_b)!}{N_c!k_b!}$ with $k_b \in \{0,\ldots,N_c\}$
- Baryon loops: l_b; ω_{l_b} depends on the loop geometry
- Quark loops: l_q; $\omega_{l_q}[\phi,\bar\phi]$ depends on the loop geometry   **NEW**

ω_{l_b} and ω_{l_q} are not positive definite. $\Rightarrow$ Still has a sign problem!

5. A first look at simulating 4d SU(3)

The relevant case: **SU(3)** and **d = 4**

Now: Consider the interesting but (more) expensive case!

Problem: The local metropolis performs worse when going to SU(3).
⇒ We need an alternative algorithm!

Possible algorithm types:

- Heatbath algorithm
  Usually shows the best performance for pure gauge theory.
- Hybrid Monte-Carlo algorithm
  The algorithm of choice if quarks should be included.

Starting point: Bosonised version

$$Z = \int [dU][d\bar{\phi}][d\phi] \, \exp\left\{-S_B[\phi, \bar{\phi}, U]\right\}$$

$$S_B[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_{p} \sum_{j=1}^{4} \Big[m\, \bar{\phi}_{b,p}(x_j)\phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_{j+1})\, U(x_{j+1}, x_j)\, \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_j)\, U(x_j, x_{j+1})\, \phi_{b,p}(x_{j+1})\Big]$$

A generic starting point

A suitable first step for all algorithms:
Draw the bosonic fields according to the distribution:

$$\exp(-S_B) = \exp(-\bar{\phi} M[U] \phi) \quad \text{with} \quad M = \text{diag}(M_p)$$

M_p are 12×12 complex matrices and can explicitly be written as $M_p = K_p^\dagger K_p$.

$\Rightarrow$ Draw fields $\eta_{b,p}$ according to $\exp(\eta_{b,p}^\dagger \eta_{b,p})$. $\rightarrow$ $\phi_{b,p} = K_p^{-1} \eta_{b,p}$

Some comments:

- The matrices K_p can be inverted explicitly and are equivalent for all b.
- With growing N_b we only need to do some additional matrix multiplications.
- In this way the role of the ϕ field is similar to the pseudo-fermionic fields for the inclusion of fermions.

Now that we obtained the fields ϕ according to the correct distribution we need to update the gauge field.

Update of the gauge field

The gauge field action is of the form:

$$S_g = \frac{1}{2}\mathrm{Tr}\left\{U\, A[\phi,\bar{\phi}] + U^\dagger\, A^\dagger[\phi,\bar{\phi}]\right\}$$

(looks similar to the Wilson action for link U with $\beta = N_c$)

Possible update algorithms:

- Use the standard SU(3) Cabbibo-Marinari heatbath for the update.
- Use the HMC algorithm to update U (if fermions are present).

Comments:

- Only the matrices A need to be stored, not the ϕ fields.
  $\Rightarrow$ Even the limit $N_b \to \infty$ is possible.
- The matrices A are constant for the HMC.
  $\Rightarrow$ No communication is needed during the MD.
- The force and action for the HMC is very easy to compute.

First tests with the HMC algorithm

Is the HMC in this form advantageous?

- It might be that scanning of the parameter space is inefficient due to the separate update of ϕ and U.
- Indeed in first tests we have seen rather large autocorrelation times.

$\Rightarrow$ Maybe it is helpfull to include an update of ϕ in the MD.

- Possible advantage: Configurations are more decorelated.
- Communication will be needed after every update of ϕ.
  $\Rightarrow$ Not problematic if we can put the ϕ update on a larger time-step.
  (in particular in combination with fermions)
- We need to store the full ϕ field.

Summary and Perspectives

- We have investigated a possible alternative discretisation of continuum pure gauge theory.
- While for $d = 2$ it can be shown that the theory has the correct continuum limit this is not guaranteed if $d > 2$.
- Numerical tests show good agreement with simulations using Wilson's gauge action, both for $T = 0$ and $T \neq 0$.
- In its original formulation with auxiliary boson fields the theory has a sign problem. $\Rightarrow$ We introduced a modified version without sign problem.
- Passing to a dual theory via a direct integration over gauge fields:
  - Leads to a theory formulated in terms of auxiliary bosonic fields.
  - When fermions are include one can expand the action in grassmann variables and integrate over the fermionic degrees of freedom and the gauge fields.
  - However, the resulting dual representation has a sign problem.
  - Is it possible to find a formulation without sign problem?
- Explore other analytical methods ...

Thank you for your attention!