Loops in SU(2) Lattice Gauge Theory

Local Dynamics on Non-intersecting Electric Flux Tubes

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Perspective and Challenges in Lattice Gauge Theory
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A Different Perspective

Hamiltonian Formulation

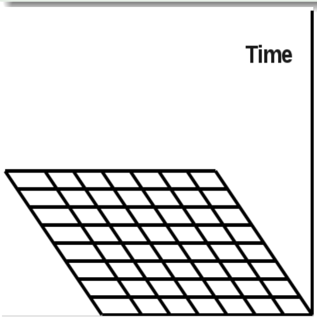
Loop Formulation

- An old problem in quantum field theory: Reformulation of gauge theories in terms of gauge invariant Wilson loops and strings carrying fluxes.
- The lattice formulation of gauge theories $\Rightarrow$ a step in this direction.
- Two major obstacles: non-locality and proliferation of loops and string states.
- Not all loop states are mutually independent $\Rightarrow$ Mandelstam constraints.
- The Mandelstam constraints, in turn, are difficult to solve because of their non-locality.
- Becomes a severe problem in the weak-coupling regime (continuum limit) of lattice gauge theory.

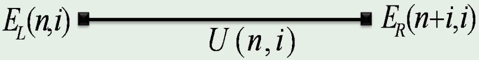
- It is important to explore new descriptions of QCD where loops, string states and their dynamics as well as the associated Mandelstam constraints can be analyzed locally.
- Prepotentials provide such a platform!

Hamiltonian LGT: Variables

Discrete Space and Continuous time



On a link of the spatial lattice



$$E_R(n+i,i) = -U^\dagger(n,i)E_L(n,i)U(n,i).$$

The Kogut-Susskind Hamiltonian

$SU(2)$ gauge theory

$$H = g^2 \sum_{n,i} \sum_{\mathrm{a}=1}^{3} E^{\mathrm{a}}(n,i) E^{\mathrm{a}}(n,i) + \frac{1}{g^2} \sum_{\Box} Tr \left(1 - U_{\Box} - U_{\Box}^{\dagger}\right)$$

with, $U_{\Box} = U(n,i)U(n+i,j)U^{\dagger}(n+j,i)U^{\dagger}(n,j)$
$\mathrm{a}(=1,2,3) \rightarrow$ **color index.**

Quantization Rules

Canonical variables

$$\begin{aligned} [E^{\mathrm{a}}_L(n,i), U^{\alpha}{}_{\beta}(n,i)] &= -\left(\frac{\sigma^{\mathrm{a}}}{2}U(n,i)\right)^{\alpha}{}_{\beta}, \\ \left[E^{\mathrm{a}}_R(n+i,i), U^{\alpha}_{\beta}(n,i)\right] &= \left(U(n,i)\frac{\sigma^{\mathrm{a}}}{2}\right)^{\alpha}{}_{\beta}. \end{aligned}$$

Constraints

Gauss Law

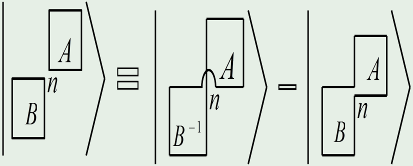
$$G(n) = \sum_{i=1}^{d} \left(E_L^{\mathrm{a}}(n,i) + E_R^{\mathrm{a}}(n,i) \right) = 0, \forall n.$$

Electric field constraint

$$E_L^2(n,i) = E_R^2(n+i,i)$$

Physical Hilbert Space: Wilson loops

Identity involving two loops, each carrying one unit of flux

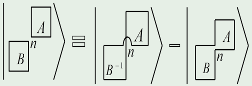


Mandelstam Identities

Wilson loops and Mandelstam Constraints: SU(2)

- **Increasing number of Loops $\Rightarrow$ Increasing number of Mandelstam Identities!**
- **But all these identities are derivable from a fundamental one!**

Fundamental Mandelstam identity for SU(2)

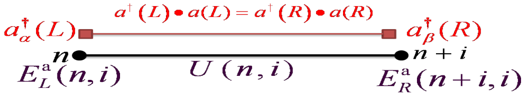


- **In prepotential formulation these fundamental Mandelstam identities becomes local and can be analyzed as well as solved to get Orthonormal Loop states.**

Prepotentials

- **Harmonic oscillators belonging to the fundamental representation of the gauge group defined at each lattice site.**
- **Prepotentials transform as matter fields $\rightarrow$ construct local gauge invariant variables and states from them!**
- **Local Mandelstam constraints $\Rightarrow$ Exact solution is possible.**
- **Prepotential formulation of SU(2), SU(3) and arbitrary SU(N) exists , but we will confine ourselves to SU(2) only in this talk.**
- *Ref: Manu Mathur, Nucl. Phys. B 2007, Phys. Letts. B 2006, J. Phys. A: Math. Gen. 2005, Ramesh Anishetty, MM, IR J. Phys. A 2010, J. Math. Phys 2010*

SU(2) Prepotentials



Left electric fields: $E^a_L(n,i) \equiv a^\dagger(n,i;L)\frac{\sigma^a}{2}a(n,i;L),$

Right electric fields: $E^a_R(n+i,i) \equiv a^\dagger(n+i,i;R)\frac{\sigma^a}{2}a(n+i,i;R).$

Under SU(2) gauge transformation

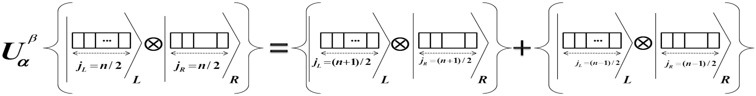
$$a^\dagger_\alpha(L) \to a^\dagger_\beta(L)\,(\Lambda^\dagger_L)^\beta{}_\alpha, \qquad a^\dagger_\alpha(R) \to a^\dagger_\beta(R)\,(\Lambda^\dagger_R)^\beta{}_\alpha$$

$$a^\alpha(L) \to (\Lambda_L)^\alpha{}_\beta\, a^\beta(L), \qquad a^\alpha(R) \to (\Lambda_R)^\alpha{}_\beta\, a^\beta(R).$$

Link Operator

- **From $SU(2) \otimes U(1)$ gauge transformations of the prepotentials,**

$$U^{\alpha}{}_{\beta} = \tilde{a}^{\dagger\alpha}(L)\,\eta\, a^{\dagger}_{\beta}(R) + a^{\alpha}(L)\,\theta\,\tilde{a}_{\beta}(R)$$



- **Calculating the coefficients from $U^{\dagger}U = UU^{\dagger} = 1$,**

$$U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}}\begin{pmatrix} a_2^{\dagger}(L) & a_1(L) \\ -a_1^{\dagger}(L) & a_2(L)\end{pmatrix}}_{U_L}\underbrace{\begin{pmatrix} a_1^{\dagger}(R) & a_2^{\dagger}(R) \\ a_2(R) & -a_1(R)\end{pmatrix}\frac{1}{\sqrt{\hat{N}+1}}}_{U_R}$$

Abelian Weaving, Non-abelian Intertwining and Loop States

Link operator: $U^{\alpha}{}_{\beta} = \frac{1}{\sqrt{\hat{n}+1}} \left(\tilde{a}^{\dagger\alpha}(L)\, a^{\dagger}_{\beta}(R) + a^{\alpha}(L)\, \tilde{a}_{\beta}(R) \right) \frac{1}{\sqrt{\hat{n}+1}}$

Four basic gauge invariant operators constructed by $U^{\alpha}{}_{\beta}(n,i)U^{\beta}{}_{\gamma}(n+i,j)$ at site $(n+i)$:

$$\hat{\mathcal{O}}^{i_+ j_+} \equiv a^{\dagger}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} \tilde{b}^{\dagger\beta}(j) = \frac{1}{\sqrt{\hat{n}_i}} \frac{1}{\sqrt{\hat{n}_j+1}} a^{\dagger}(i)\cdot\tilde{b}^{\dagger\beta}(j) \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} k^{ij}_{+}$$

$$\hat{\mathcal{O}}^{i_+ j_-} \equiv a^{\dagger}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} b^{\beta}(j) = \frac{1}{\sqrt{\hat{n}_i}} a^{\dagger}(i)\cdot b(j) \frac{1}{\sqrt{(\hat{n}_j+2)}} \equiv \frac{1}{\sqrt{\hat{n}_i}} \kappa^{ij} \frac{1}{\sqrt{(\hat{n}_j+2)}}$$

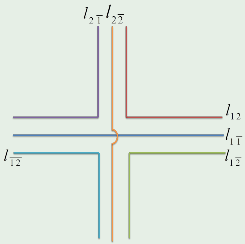
$$\hat{\mathcal{O}}^{j_+ i_-} \equiv \tilde{a}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} \tilde{b}^{\dagger\beta}(j) = \frac{1}{\sqrt{(\hat{n}_j+1)}} a(i)\cdot b^{\dagger}(j) \frac{1}{\sqrt{n_i+1}} \equiv \frac{1}{\sqrt{(\hat{n}_j+1)}} \kappa^{ji} \frac{1}{\sqrt{n_i+1}}$$

$$\hat{\mathcal{O}}^{i_- j_-} \equiv \tilde{a}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} b^{\beta}(j) = \tilde{a}(i)\cdot b(j) \frac{1}{\sqrt{(\hat{n}_i+1)(\hat{n}_j+2)}} \equiv k^{ji}_{-} \frac{1}{\sqrt{(\hat{n}_i+1)(\hat{n}_j+2)}}$$

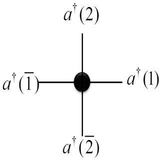
Loop States and Linking Numbers

$$|\{l_{ij}\}\rangle = \prod_{i\neq j} \frac{(k_+)^{l_{ij}}}{l_{ij}!}|0\rangle$$

Linking numbers in 2d

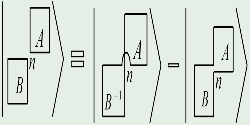


Mandelstam Constraints



$$\left(a^\dagger(1)\cdot\tilde{a}^\dagger(2)\right)\left(a^\dagger(\bar{1})\cdot\tilde{a}^\dagger(\bar{2})\right)\equiv\left(a^\dagger(1)\cdot\tilde{a}^\dagger(\bar{1})\right)\left(a^\dagger(2)\cdot\tilde{a}^\dagger(\bar{2})\right)-\left(a^\dagger(1)\cdot\tilde{a}^\dagger(\bar{2})\right)\left(a^\dagger(2)\cdot\tilde{a}^\dagger(\bar{1})\right)$$

Equivalen to the fundamental Mandelstam identity



Linking Numbers and Constraints

Loop State characterized by 6 linking numbers

$$|l_{12}, l_{1\bar{1}}, l_{1\bar{2}}, l_{2\bar{1}}, l_{2\bar{2}}, l_{\bar{1}\bar{2}}\rangle \equiv |\{l\}\rangle = \frac{\left(k_+^{12}\right)^{l_{12}}}{l_{12}!} \frac{\left(k_+^{1\bar{1}}\right)^{l_{1\bar{1}}}}{l_{1\bar{1}}!} \frac{\left(k_+^{1\bar{2}}\right)^{l_{1\bar{2}}}}{l_{1\bar{2}}!} \frac{\left(k_+^{2\bar{1}}\right)^{l_{2\bar{1}}}}{l_{2\bar{1}}!} \frac{\left(k_+^{2\bar{2}}\right)^{l_{2\bar{2}}}}{l_{2\bar{2}}!} \frac{\left(k_+^{\bar{1}\bar{2}}\right)^{l_{\bar{1}\bar{2}}}}{l_{\bar{1}\bar{2}}!} |0\rangle \qquad (1)$$

with $n_1 = l_{12} + l_{1\bar{1}} + l_{1\bar{2}}$, $n_2 = l_{2\bar{1}} + l_{2\bar{2}} + l_{12}$, $n_{\bar{1}} = l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{1}}$, $n_{\bar{2}} = l_{1\bar{2}} + l_{2\bar{2}} + l_{\bar{1}\bar{2}}$

One Mandelstam constraint

$$k_+^{12}k_+^{\bar{1}\bar{2}} - k_+^{1\bar{2}}k_+^{2\bar{1}} + k_+^{1\bar{1}}k_+^{2\bar{2}} = 0$$

Two $U(1)$ Gauss Law constraints

$$n_1(x) = n_{\bar{1}}(x + e_1) \;\;\&\;\; n_2(x) = n_{\bar{2}}(x + e_2)$$

Three different class of loop operators

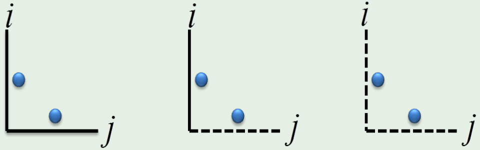
$$\hat{\mathcal{O}}^{i_+ j_+} \equiv \frac{1}{\sqrt{(n_i+1)(n_j+2)}} k_+^{ij}$$

$$\hat{\mathcal{O}}^{i_+ j_-} \equiv \frac{1}{\sqrt{(n_i+1)(n_j+2)}} \kappa^{ij}$$

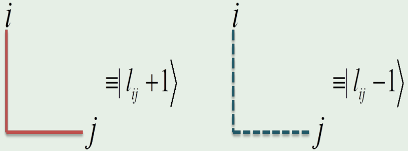
$$\hat{\mathcal{O}}^{i_- j_-} \equiv \frac{1}{\sqrt{(n_i+1)(n_j+2)}} k_-^{ij}$$

Pictorial Representation

Local Loop Operators



Action on States

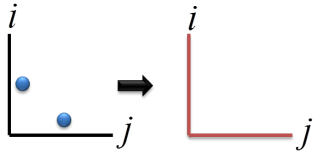


Local Action of Loop Operators on Local Loop states

Action of $\hat{\mathcal{O}}^{i_+ j_+}$

$$\hat{\mathcal{O}}^{i_+ j_+} |\{l\}\rangle \equiv \frac{1}{\sqrt{(n_i+1)(n_j+2)}} \left(a^\dagger(i) \cdot \tilde{b}^\dagger(j)\right) |\{l\}\rangle = \frac{(l_{ij}+1)}{\sqrt{(n_i+1)(n_j+2)}} |l_{ij}+1\rangle$$

or pictorially:

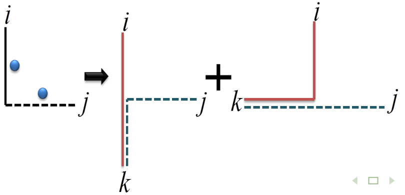


Local Action of Loop Operators on Local Loop states

Action of $\hat{\mathcal{O}}^{i+j-}$

$$
\begin{aligned}
\hat{\mathcal{O}}^{i+j-}\,|\{l\}\rangle &\equiv \frac{1}{\sqrt{(n_i+1)(n_j+2)}}\left(a^{\dagger}(i)\cdot b(j)\right)|\{l\}\rangle \\
&= \frac{1}{\sqrt{(n_i+1)(n_j+2)}}\sum_{k\neq i,j}(-1)^{S_{ik}}(l_{ik}+1)|l_{jk}-1,\, l_{ik}+1\rangle
\end{aligned}
$$

where, in l_{ij} the first index is always less than the second one with ordering convention $1 < 2 < \bar{1} < \bar{2}$ and $S_{ik}=1$ if $i>k$ & $S_{ik}=0$ if $i<k$.



Local Action of Loop Operators on Local Loop states

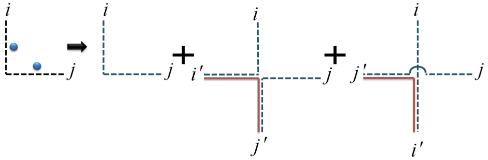
Action of $\hat{\mathcal{O}}^{i-j-}$

$$\hat{\mathcal{O}}^{i-j-}\,|\{l\}\rangle = \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}}\left(a(i)\cdot\tilde{b}(j)\right)|\{l\}\rangle$$

$$= \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}}\left[(n_i+n_j-l_{ij}+1)|l_{ij}-1\rangle + \sum_{\bar{i},\bar{j}\{\neq i,j\}}(l_{\bar{i}\bar{j}}+1)(-1)^{S_{\bar{i}\bar{j}}}|l_{i\bar{i}}-1, l_{j\bar{j}}-1, l_{\bar{i}\bar{j}}+1\rangle\right]$$

with $S_{ik}=1$ if $i>k$ & $S_{ik}=0$ if $i<k$.

Pictorially,

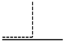


Diagrametic rules

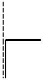
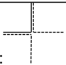
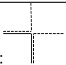
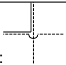
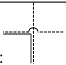
Rules to calculate the coefficient from a loop diagram:

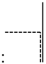
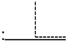
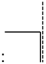
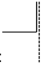
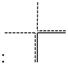
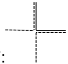
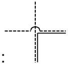
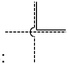
- Any diagram with net flux increasing or decreasing along $i - j$ direction (or increasing along i and decreasing along j directions together) contribute a factor of $\frac{1}{\sqrt{(n_i+1)(n_j+2)}}$, where n_i, n_j counts the flux of the state on which the loop operator has acted.
- Each solid line crossing the site from direction $i - j$ will contribute a factor of $l_{ij} + 1$.
- Each dotted line crossing the site from direction $i - j$, without having any overlap with any solid line on any of its arm, will contribute a factor of $(n_i + n_j - l_{ij} + 1)$.
- Each solid flux line along $i - k$ direction with the link at k direction, having overlap with a dotted link along $k - j$ direction will contribute a factor of $(-1)^{S_{ik}}$.
- Each solid flux line along $i - j$ direction with the link at i direction, having overlap with a dotted link along $i - i'$ direction and the link at j direction, having overlap with a dotted link along $j - j'$ direction will contribute a factor of $(-1)^{S_{ij}}$.

Illustrating the diagrammatic schemes:

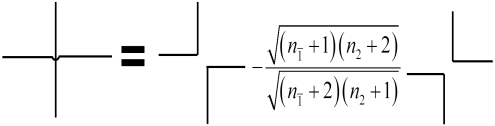
vertex	coefficient	vertex	coefficient
d1: 	$C^{1+\bar{2}+} = \frac{l_{1\bar{2}}+1}{\sqrt{(n_1+1)(n_{\bar{2}}+2)}}$	a1: 	$C^{1+2+} = \frac{l_{12}+1}{\sqrt{(n_1+1)(n_2+2)}}$
d2: 	$C^{1-\bar{2}-} = \frac{(n_1+n_{\bar{2}}-l_{1\bar{2}}+1)}{\sqrt{(n_1+1)(n_{\bar{2}}+2)}}$	a2: 	$C^{1-2-} = \frac{(n_1+n_2-l_{12}+1)}{\sqrt{(n_1+1)(n_2+2)}}$
d3: 	$\left(C^{\bar{2}+1-}\right)_{\bar{1}} = -\frac{l_{\bar{1}\bar{2}}+1}{\sqrt{(n_{\bar{2}}+1)(n_1+2)}}$	a3: 	$\left(C^{2+1-}\right)_{\bar{1}} = \frac{l_{2\bar{1}}+1}{\sqrt{(n_2+1)(n_1+2)}}$
d4: 	$\left(C^{\bar{2}+1-}\right)_{2} = -\frac{l_{2\bar{2}}+1}{\sqrt{(n_{\bar{2}}+1)(n_1+2)}}$	a4: 	$\left(C^{2+1-}\right)_{\bar{2}} = \frac{l_{2\bar{2}}+1}{\sqrt{(n_2+1)(n_1+2)}}$
d5: 	$\left(C^{1+\bar{2}-}\right)_{\bar{1}} = \frac{l_{1\bar{1}}+1}{\sqrt{(n_1+1)(n_{\bar{2}}+2)}}$	a5: 	$\left(C^{1+2-}\right)_{\bar{1}} = \frac{l_{1\bar{1}}+1}{\sqrt{(n_1+1)(n_2+2)}}$

Illustrating the diagrammatic schemes:

vertex	coefficient	vertex	coefficient
d6: 	$\left(C^{1+\bar{2}-}\right)_2 = \frac{l_{12}+1}{\sqrt{(n_1+1)(n_{\bar 2}+2)}}$	a6: 	$\left(C^{1+2-}\right)_{\bar 2} = \frac{l_{1\bar 2}+1}{\sqrt{(n_1+1)(n_2+2)}}$
d7: 	$C^{(1-)_2(\bar 2-)_{\bar 1}} = \frac{l_{2\bar 1}+1}{\sqrt{(n_1+1)(n_{\bar 2}+2)}}$	a7: 	$C^{(1-)_{\bar 2}(2-)_{\bar 1}} = -\frac{l_{\bar 1\bar 2}+1}{\sqrt{(n_1+1)(n_2+2)}}$
d8: 	$C^{(1-)_{\bar 1}(\bar 2-)_2} = -\frac{l_{2\bar 1}+1}{\sqrt{(n_1+1)(n_{\bar 2}+2)}}$	a8: 	$C^{(1-)_{\bar 1}(2-)_{\bar 2}} = \frac{l_{\bar 1\bar 2}+1}{\sqrt{(n_1+1)(n_2+2)}}$
b1: 	$C^{\bar 1+2+} = \frac{l_{\bar 1 2}+1}{\sqrt{(n_{\bar 1}+1)(n_2+2)}}$	c1: 	$C^{\bar 1+\bar 2+} = \frac{l_{\bar 1\bar 2}+1}{\sqrt{(n_{\bar 1}+1)(n_{\bar 2}+2)}}$
b2: 	$C^{\bar 1-2-} = \frac{(n_{\bar 1}+n_2-l_{\bar 1 2}+1)}{\sqrt{(n_{\bar 1}+1)(n_2+2)}}$	c2: 	$C^{\bar 1-\bar 2-} = \frac{(n_{\bar 1}+n_{\bar 2}-l_{\bar 1\bar 2}+1)}{\sqrt{(n_{\bar 1}+1)(n_{\bar 2}+2)}}$

vertex	coefficient	vertex	coefficient
b3: 	$\left(C^{2+\bar{1}-}\right)_1 = -\frac{l_{12}+1}{\sqrt{(n_2+1)(n_{\bar{1}}+2)}}$	c3: 	$\left(C^{\bar{2}+\bar{1}-}\right)_1 = -\frac{l_{1\bar{2}}+1}{\sqrt{(n_{\bar{2}}+1)(n_{\bar{1}}+2)}}$
b4: 	$\left(C^{2+\bar{1}-}\right)_{\bar{2}} = \frac{l_{2\bar{2}}+1}{\sqrt{(n_2+1)(n_{\bar{1}}+2)}}$	c4: 	$\left(C^{\bar{2}+\bar{1}-}\right)_2 = -\frac{l_{2\bar{2}}+1}{\sqrt{(n_{\bar{2}}+1)(n_{\bar{1}}+2)}}$
b5: 	$\left(C^{\bar{1}+2-}\right)_1 = -\frac{l_{1\bar{1}}+1}{\sqrt{(n_{\bar{1}}+1)(n_2+2)}}$	c5: 	$\left(C^{\bar{1}+\bar{2}-}\right)_1 = -\frac{l_{1\bar{1}}+1}{\sqrt{(n_{\bar{1}}+1)(n_{\bar{2}}+2)}}$
b6: 	$\left(C^{\bar{1}+2-}\right)_{\bar{2}} = \frac{l_{\bar{1}\bar{2}}+1}{\sqrt{(n_{\bar{1}}+1)(n_2+2)}}$	c6: 	$\left(C^{\bar{1}+\bar{2}-}\right)_2 = -\frac{l_{2\bar{1}}+1}{\sqrt{(n_{\bar{1}}+1)(n_{\bar{2}}+2)}}$
b7: 	$C^{(2-)_1(\bar{1}-)_{\bar{2}}} = \frac{l_{1\bar{2}}+1}{\sqrt{(n_{\bar{1}}+1)(n_2+2)}}$	c7: 	$C^{(\bar{1}-)_2(\bar{2}-)_1} = -\frac{l_{12}+1}{\sqrt{(n_{\bar{1}}+1)(n_{\bar{2}}+2)}}$
b8: 	$C^{(2-)_{\bar{2}}(\bar{1}-)_1} = -\frac{l_{1\bar{2}}+1}{\sqrt{(n_{\bar{1}}+1)(n_2+2)}}$	c8: 	$C^{(\bar{1}-)_1(\bar{2}-)_2} = \frac{l_{12}+1}{\sqrt{(n_{\bar{1}}+1)(n_{\bar{2}}+2)}}$

Mandelstam Constraint $k_+^{12}k_+^{\bar{1}\bar{2}} - k_+^{1\bar{2}}k_+^{2\bar{1}} + k_+^{1\bar{1}}k_+^{2\bar{2}} = 0$



Consider our physical loop Hilbert space to solve the Mandelstam constraint by:

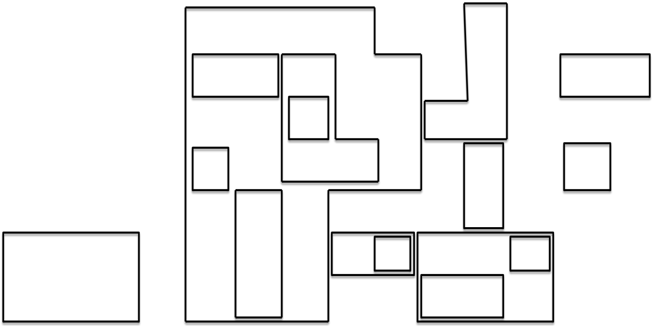
$$l_{1\bar{1}}(x) \cdot l_{2\bar{2}}(x) = 0$$

$$\Downarrow$$

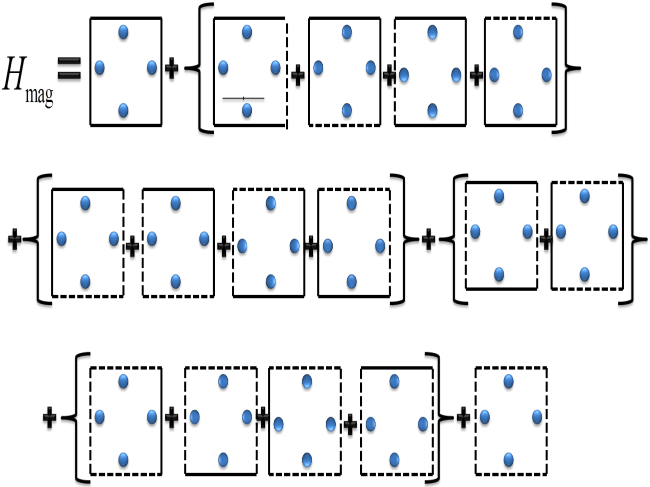
Only non-intersecting loops are physical

Vertices a8, b8, c8, d8 vanish on physical loop space!

Physical Loop



Magnetic Hamiltonian in prepotential formulation



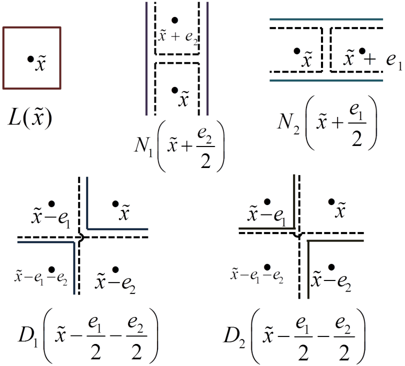
Loop dynamics and linking numbers

- Linking numbers $\Rightarrow$ ultra local description of loop states.
- Hamiltonian dynamics is always around a plaquette.
- Abelian gauss law constraints need to be imposed.

$$\Downarrow$$

Alternate description of loop states by a set of Fusion Variables.

Fusion Variables

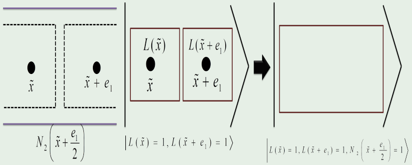


Any Loop can be characterized by

$$|LS\rangle \equiv \prod_x |L, N_1, N_2, D_1, D_2\rangle_x$$

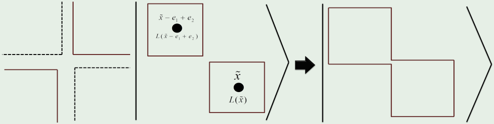
action of the fusion operators

$$\Pi^{+}_{N_2}(\tilde{x}+\frac{e_1}{2})|L(\tilde{x})=1,L(\tilde{x}+e_1)=1\rangle=|L(\tilde{x})=1,L(\tilde{x}+e_1)=1,N_2(\tilde{x}+\frac{e_1}{2})=1\rangle$$



action of the fusion operators

$$\Pi^-_{D_2}\Pi^+_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})|L(\tilde{x})=1, L(\tilde{x}-e_1+e_2)=1\rangle = |L(\tilde{x})=1, L(\tilde{x}+e_1)=1, D_1(\tilde{x}-\frac{e_1}{2}=1, D_2(\tilde{x}-\frac{e_1}{2}=-$$



Fusion Variables and Linking Numbers

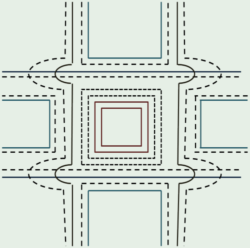
$$
\begin{aligned}
l_{12}(x) &= L(\tilde{x}) - N_2(\tilde{x} - \frac{e_1}{2}) - N_1(\tilde{x} - \frac{e_2}{2}) + D_1(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{1\bar{1}}(x) &= N_2(\tilde{x} - \frac{e_1}{2}) + N_2(\tilde{x} - \frac{e_1}{2} - e_2) - D_1(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) - D_2(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{1\bar{2}}(x) &= L(\tilde{x} - e_2) - N_2(\tilde{x} - \frac{e_1}{2} - e_2) - N_1(\tilde{x} - \frac{e_2}{2}) + D_2(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{2\bar{1}}(x) &= L(\tilde{x} - e_1) - N_2(\tilde{x} - \frac{e_1}{2}) - N_1(\tilde{x} - e_1 - \frac{e_2}{2}) + D_2(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{2\bar{2}}(x) &= N_1(\tilde{x} - \frac{e_2}{2}) + N_1(\tilde{x} - e_1 - \frac{e_2}{2}) - D_1(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) - D_2(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{\bar{1}\bar{2}}(x) &= L(\tilde{x} - e_1 - e_2) - N_2(\tilde{x} - \frac{e_1}{2} - e_2) - N_1(\tilde{x} - e_1 - \frac{e_2}{2}) + D_1(\tilde{x} - \frac{e_1}{2} - - \frac{e_2}{2})
\end{aligned}
$$

The Shift Operators corresponding to Fusion Variables

$$\begin{aligned}
\hat{L}(\tilde{x})\Pi^{\pm}_{L}(\tilde{x})|L, N_1, N_2, D_1, D_2\rangle &= (L(\tilde{x})\pm 1)|L, N_1, N_2, D_1, D_2\rangle \\
\hat{N}_1(\tilde{x}-\frac{e_2}{2})\Pi^{\pm}_{N_1}(\tilde{x}-\frac{e_2}{2})|L, N_1, N_2, D_1, D_2\rangle &= (N_1(\tilde{x}-\frac{e_2}{2})\pm 1)|L, N_1, N_2, D_1, D_2\rangle \\
\hat{N}_2(\tilde{x}-\frac{e_1}{2})\Pi^{\pm}_{N_2}(\tilde{x}-\frac{e_1}{2})|L, N_1, N_2, D_1, D_2\rangle &= (N_2(\tilde{x}-\frac{e_1}{2})\pm 1)|L, N_1, N_2, D_1, D_2\rangle \\
\hat{D}_1(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^{\pm}_{D_1}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})|L, N_1, N_2, D_1, D_2\rangle &= (D_1(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\pm 1)|L, N_1, N_2, D_1, D_2\rangle \\
\hat{D}_2(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^{\pm}_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})|L, N_1, N_2, D_1, D_2\rangle &= (D_2(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\pm 1)|L, N_1, N_2, D_1, D_2\rangle
\end{aligned}$$

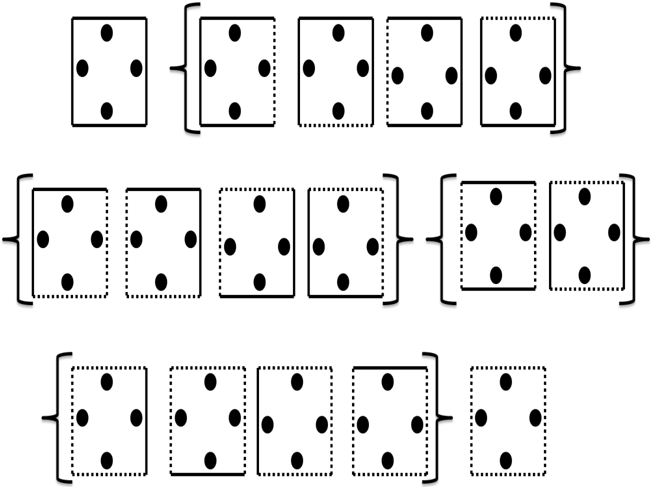
Fusion Variables and Constraints

5 Variables-1Mandelstam Constranit⇒ Another additional Constraint



$$\Pi^-_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^-_{D_2}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_1}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})$$
$$\Pi^+_{N_1}(\tilde{x}-\frac{e_2}{2})\Pi^+_{N_1}(\tilde{x}+\frac{e_2}{2})\Pi^+_{N_2}(\tilde{x}-\frac{e_1}{2})\Pi^+_{N_2}(\tilde{x}+\frac{e_1}{2})\left(\Pi^+_L(\tilde{x})\right)^2=1$$

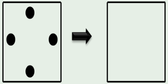
Magnetic Hamiltonian: 16 plaquette operators



Action of $\hat{H}_{mag}$ on Loop States

Type a: H_{++++}

Explicit action: 1 state

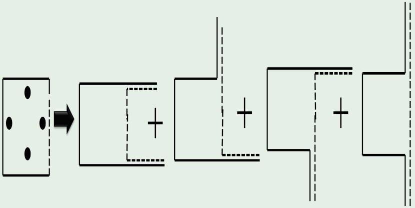


Explicit Realization in terms of Fusion Operators

$$H_1 = C_a^{1_+ 2_+} C_b^{2_+ \bar{1}_+} C_c^{\bar{1}_+ \bar{2}_+} C_d^{1_+ \bar{2}_+} \Pi_L^+(\tilde{x})$$

Action of $\hat{H}_{mag}$ on Loop States

Type b: H_{+++-}



Explicit action $\Rightarrow$ 4 states

Explicit Realization in terms of Fusion Operators

Type b: H_{+++-}

$$H_2 = C_a^{1+2+} C_d^{1+\bar{2}+} \left((C_c^{\bar{1}+\bar{2}-})_1 + (C_c^{\bar{1}+\bar{2}-})_2 \Pi^+_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2}) \right)$$
$$\left((C_b^{2-\bar{1}+})_1 + (C_b^{2-\bar{1}+})_{\bar{2}} \Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2}) \right) \Pi^+_{N_2}(\tilde{x} + \frac{e_1}{2}) \Pi^+_L(\tilde{x}) \quad (2)$$

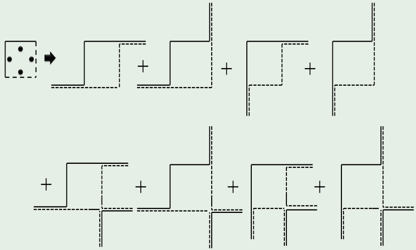
$$H_3 = C_c^{\bar{1}+\bar{2}+} C_d^{1+\bar{2}+} \left((C_a^{1-2+})_{\bar{2}} + (C_a^{1-2+})_{\bar{1}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \right)$$
$$\left((C_b^{2+\bar{1}-})_{\bar{2}} + (C_b^{2+\bar{1}-})_1 \Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2}) \right) \Pi^+_{N_1}(\tilde{x} - \frac{e_2}{2}) \Pi^+_L(\tilde{x}) \quad (3)$$

$$H_4 = C_b^{2+\bar{1}+} C_c^{\bar{1}+\bar{2}+} \left((C_a^{1+2-})_{\bar{1}} + (C_a^{1+2-})_{\bar{2}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \right)$$
$$\left((C_d^{1+\bar{2}-})_{\bar{1}} + (C_d^{1+\bar{2}-})_2 \Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2}) \right) \Pi^+_{N_2}(\tilde{x} - \frac{e_1}{2}) \Pi^+_L(\tilde{x}) \quad (4)$$

$$H_5 = C_a^{1+2+} C_b^{2+\bar{1}+} \left((C_c^{\bar{1}-\bar{2}+})_2 + (C_c^{\bar{1}-\bar{2}+})_1 \Pi^+_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2}) \right)$$
$$\left((C_d^{1-\bar{2}+})_2 + (C_d^{1-\bar{2}+})_{\bar{1}} \Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2}) \right) \Pi^+_{N_1}(\tilde{x} + \frac{e_2}{2}) \Pi^+_L(\tilde{x}) \quad (5)$$

Action of $\hat{H}_{mag}$ on Loop States

Type c: H_{++--}



The explicit action $1 \times 2^3 = 8$ states

Explicit Realization in terms of Fusion Operators

Type c: H_{++--}

$$
\begin{aligned}
H_6 \quad = \quad & \left(C_d^{1+\bar{2}+}\Pi^-_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\right)\left((C_c^{\bar{1}+\bar{2}-})_1+(C_c^{\bar{1}+\bar{2}-})_2\Pi^-_{D_2}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\right) \\
& \left((C_a^{1-2+})_{\bar{2}}+(C_a^{1-2+})_{\bar{1}}\Pi^-_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\right) \qquad (6) \\
& \left(C_b^{2-\bar{1}-}+C_b^{(2-)_1(\bar{1}-)_{\bar{2}}}\Pi^+_{D_2}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})\Pi^-_{D_1}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})\right)\Pi^-_{N_2}(\tilde{x}-\frac{e_1}{2})\Pi^-_{N_1}(\tilde{x}+\frac{e_2}{2})\Pi^-_L(\tilde{x}) \\
H_7 \quad = \quad & \left(C_c^{\bar{1}+\bar{2}+}\Pi^-_{D_2}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\right)\left((C_b^{2+\bar{1}-})_{\bar{2}}+(C_b^{2+\bar{1}-})_1\Pi^-_{D_1}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})\right) \\
& \left((C_d^{1+\bar{2}-})_{\bar{1}}+(C_d^{1+\bar{2}-})_2\Pi^-_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\right) \qquad (7) \\
& \left(C_a^{1-2-}+C_a^{(1-)_{\bar{2}}(2-)_{\bar{1}}}\Pi^+_{D_1}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^-_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\right)\Pi^-_{N_2}(\tilde{x}+\frac{e_1}{2})\Pi^-_{N_1}(\tilde{x}+\frac{e_2}{2})\Pi^-_L(\tilde{x})
\end{aligned}
$$

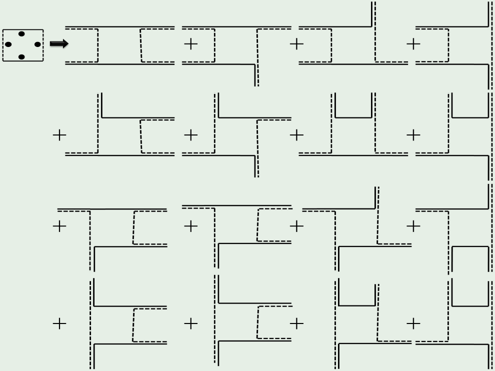
Explicit Realization in terms of Fusion Operators

Type c: H_{++--}

$$
\begin{aligned}
H_8 \quad = \quad & \left(C_b^{2+\bar{1}+}\Pi^-_{D_1}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})\right)\left((C_c^{\bar{1}+\bar{2}-})_1+(C_c^{\bar{1}+\bar{2}-})_2\Pi^-_{D_2}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\right) \\
& \left((C_a^{1-2+})_{\bar{2}}+(C_a^{1-2+})_{\bar{1}}\Pi^-_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\right) \qquad (8) \\
& \left(C_d^{1-\bar{2}-}+C_d^{(1-)_2(\bar{2}-)_{\bar{1}}}\Pi^+_{D_2}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\right)\Pi^-_{N_2}(\tilde{x}+\frac{e_1}{2})\Pi^-_{N_1}(\tilde{x}-\frac{e_2}{2})\Pi^-_L(\tilde{x}) \\
H_9 \quad = \quad & \left(C_a^{1+2+}\Pi^-_{D_2}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\right)\left((C_d^{1+\bar{2}-})_{\bar{1}}+(C_d^{1+\bar{2}-})_2\Pi^-_{D_1}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\right) \\
& \left((C_b^{2+\bar{1}-})_{\bar{2}}+(C_b^{2+\bar{1}-})_1\Pi^-_{D_1}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})\right) \qquad (9) \\
& \left(C_c^{\bar{1}-\bar{2}-}+C_c^{(\bar{1}-)_2(\bar{2}-)_1}\Pi^+_{D_1}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_2}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\right)\Pi^-_{N_2}(\tilde{x}-\frac{e_1}{2})\Pi^-_{N_1}(\tilde{x}-\frac{e_2}{2})\Pi^-_L(\tilde{x})
\end{aligned}
$$

Action of $\hat{H}_{mag}$ on Loop States

Type d: H_{+-+-}



The explicit action $\Rightarrow 2^4 = 16$ states

Explicit Realization in terms of Fusion Operators

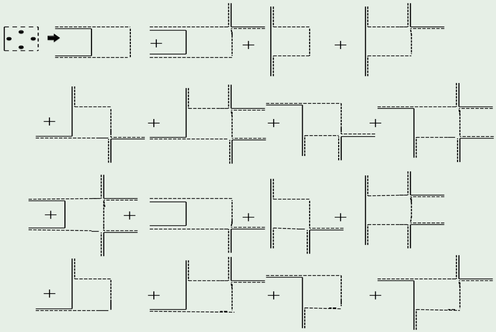
Type d: H_{+-+-}

Type (d):

$$
\begin{aligned}
H_{10} = {} & \left((C_c^{\bar{1}+\bar{2}-})_1 + (C_c^{\bar{1}+\bar{2}-})_2 \Pi^+_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\right) \\
& \left((C_b^{2-\bar{1}+})_1 + (C_b^{2-\bar{1}+})_{\bar{2}} \Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left((C_a^{1+2-})_{\bar{1}} + (C_a^{1+2-})_{\bar{2}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left((C_d^{1+\bar{2}-})_{\bar{1}} + (C_d^{1+\bar{2}-})_2 \Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\right) \Pi^+_{N_2}(\tilde{x} + \frac{e_1}{2}) \Pi^+_{N_2}(\tilde{x} - \frac{e_1}{2}) \Pi^+_L(\tilde{x}) \quad (10) \\
H_{11} = {} & \left((C_a^{1-2+})_{\bar{2}} + (C_a^{1-2+})_{\bar{1}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left((C_b^{2+\bar{1}-})_{\bar{2}} + (C_b^{2+\bar{1}-})_1 \Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left((C_c^{\bar{1}-\bar{2}+})_2 + (C_c^{\bar{1}-\bar{2}+})_1 \Pi^+_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\right) \\
& \left((C_d^{1-\bar{2}+})_2 + (C_d^{1-\bar{2}+})_{\bar{1}} \Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\right) \Pi^+_{N_1}(\tilde{x} + \frac{e_2}{2}) \Pi^+_{N_1}(\tilde{x} - \frac{e_2}{2}) \Pi^+_L(\tilde{x}) \quad (11)
\end{aligned}
$$

Action of $\hat{H}_{mag}$ on Loop States

Type e: H_{+---}



The explicit action $2^4 = 16$ states

Explicit Realization in terms of Fusion Operators

Type e: H_{---+}

$$
\begin{aligned}
H_{12} \quad = \quad & \left((C_c^{\bar{1}+\bar{2}-})_1 + (C_c^{\bar{1}+\bar{2}-})_2 \Pi^-_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\right) \\
& \left((C_b^{2+\bar{1}-})_{\bar{2}} + (C_b^{2+\bar{1}-})_1 \Pi^-_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left(C_a^{1-2-} + C_a^{(1-)_{\bar{2}}(2-)_{\bar{1}}} \Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \Pi^-_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\right) \qquad (12) \\
& \left(C_d^{1-\bar{2}-} + C_d^{(1-)_2(\bar{2}-)_{\bar{1}}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2}) \Pi^-_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\right) \Pi^-_{N_2}(\tilde{x} + \frac{e_1}{2}) \Pi^-_{L}(\tilde{x}) \\
H_{13} \quad = \quad & \left((C_a^{1-2+})_{\bar{2}} + (C_a^{1-2+})_{\bar{1}} \Pi^-_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left((C_b^{2+\bar{1}-})_{\bar{2}} + (C_b^{2+\bar{1}-})_1 \Pi^-_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left(C_c^{\bar{1}-\bar{2}-} + C_c^{(\bar{1}-)_2(\bar{2}-)_1} \Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2}) \Pi^-_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\right) \\
& \left(C_d^{1-\bar{2}-} + C_d^{(1-)_2(\bar{2}-)_{\bar{1}}} \Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2}) \Pi^-_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\right) \Pi^-_{N_1}(\tilde{x} - \frac{e_2}{2}) \Pi^-_{L}(\tilde{x}) \qquad (13)
\end{aligned}
$$

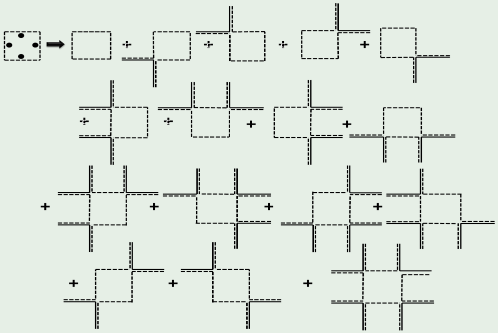
Explicit Realization in terms of Fusion Operators

Type e: H_{---+}

$$
\begin{aligned}
H_{14} \;=\; & \left((C_a^{1-2+})_{\bar 2} + (C_a^{1-2+})_{\bar 1}\Pi^-_{D_2}(\tilde x - \frac{e_1}{2} - \frac{e_2}{2}) \right) \\
& \left((C_d^{1+\bar 2-})_{\bar 1} + (C_d^{1+\bar 2-})_2\Pi^-_{D_1}(\tilde x - \frac{e_1}{2} + \frac{e_2}{2}) \right) \\
& \left(C_b^{2-\bar 1-} + C_b^{(2-)_1(\bar 1-)_{\bar 2}}\Pi^+_{D_2}(\tilde x + \frac{e_1}{2} - \frac{e_2}{2})\Pi^-_{D_1}(\tilde x + \frac{e_1}{2} - \frac{e_2}{2}) \right) \\
& \left(C_c^{\bar 1-\bar 2-} + C_c^{(\bar 1-)_2(\bar 2-)_1}\Pi^+_{D_1}(\tilde x + \frac{e_1}{2} + \frac{e_2}{2})\Pi^-_{D_2}(\tilde x - \frac{e_1}{2} + \frac{e_2}{2}) \right) \Pi^-_{N_2}(\tilde x - \frac{e_1}{2})\Pi^-_{L}(\tilde x) \quad (14) \\
H_{15} \;=\; & \left((C_c^{\bar 1+\bar 2-})_1 + (C_c^{\bar 1+\bar 2-})_2\Pi^-_{D_2}(\tilde x + \frac{e_1}{2} + \frac{e_2}{2}) \right) \\
& \left((C_d^{1+\bar 2-})_{\bar 1} + (C_d^{1+\bar 2-})_2\Pi^-_{D_1}(\tilde x - \frac{e_1}{2} + \frac{e_2}{2}) \right) \\
& \left(C_a^{1-2-} + C_a^{(1-)_{\bar 2}(2-)_{\bar 1}}\Pi^+_{D_1}(\tilde x - \frac{e_1}{2} - \frac{e_2}{2})\Pi^-_{D_2}(\tilde x - \frac{e_1}{2} - \frac{e_2}{2}) \right) \\
& \left(C_b^{2-\bar 1-} + C_b^{(2-)_1(\bar 1-)_{\bar 2}}\Pi^+_{D_2}(\tilde x + \frac{e_1}{2} - \frac{e_2}{2})\Pi^-_{D_1}(\tilde x + \frac{e_1}{2} - \frac{e_2}{2}) \right) \Pi^-_{N_1}(\tilde x + \frac{e_2}{2})\Pi^-_{L}(\tilde x) \quad (15)
\end{aligned}
$$

Action of $\hat{H}_{mag}$ on Loop States

Type f: H_{----}



The explicit action $\Rightarrow$ 16 states

Explicit Realization in terms of Fusion Operators

Type e: H_{----}

$$
\begin{aligned}
H_{16} \quad = \quad & \left(C_a^{1-2-} + C_a^{(1-)\bar{2}(2-)\bar{1}}\Pi^+_{D_1}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\Pi^-_{D_2}(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left(C_b^{2-\bar{1}-} + C_b^{(2-)1(\bar{1}-)\bar{2}}\Pi^+_{D_2}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\Pi^-_{D_1}(\tilde{x} + \frac{e_1}{2} - \frac{e_2}{2})\right) \\
& \left(C_c^{\bar{1}-\bar{2}-} + C_c^{(\bar{1}-)2(\bar{2}-)1}\Pi^+_{D_1}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\Pi^-_{D_2}(\tilde{x} + \frac{e_1}{2} + \frac{e_2}{2})\right) \\
& \left(C_d^{1-\bar{2}-} + C_d^{(1-)2(\bar{2}-)\bar{1}}\Pi^+_{D_2}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\Pi^-_{D_1}(\tilde{x} - \frac{e_1}{2} + \frac{e_2}{2})\right)\Pi^-_L(\tilde{x}) \qquad (16)
\end{aligned}
$$

- To compute physical results using this formalism, one should first calculate the norm.

- To compute physical results using this formalism, one should first calculate the norm.
- We compute the norm of loop states by noticing that this is itself product of the norms defined at the each sites of a loop state.

- To compute physical results using this formalism, one should first calculate the norm.
- We compute the norm of loop states by noticing that this is itself product of the norms defined at the each sites of a loop state.
- Next, we briefly illustrate how the strong coupling series in this new formalism, using the lattice Feynmann rules prescribed in this work matches exactly with the conventional approach. Note that, our formulation is much more simple as there is no need to deal with any complex 6j coeffiecient and is well suited for numerical computation.

Normalization of local loop states

Local loop state: $|l_{12}, l_{1\bar{1}}, l_{1\bar{2}}, l_{2\bar{1}}, l_{2\bar{2}}, l_{\bar{1}\bar{2}}\rangle$

Consider the norm of a special case: $\langle l'_{12} = 0 | l_{12} = 0\rangle$

$$
\begin{aligned}
&= \frac{(l_{1\bar{2}} + l_{2\bar{1}} + l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{2}} + 1)!}{l_{1\bar{2}}!\,(l_{2\bar{1}} + l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{2}} + 1)!}\delta_{l'_{1\bar{2}},l_{1\bar{2}}} \\
&\quad \times \frac{(l_{2\bar{1}} + l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{2}} + 1)!}{l_{2\bar{1}}!\,(l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{2}} + 1)!}\delta_{l'_{2\bar{1}},l_{2\bar{1}}} \\
&\quad \times \frac{(l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{2}} + 1)!}{l_{\bar{1}\bar{2}}!\,(l_{1\bar{1}} + l_{2\bar{2}} + 1)!}\delta_{l'_{\bar{1}\bar{2}},l_{\bar{1}\bar{2}}} \\
&\quad \times (l_{1\bar{1}} + 1)\,(l_{2\bar{2}} + 1)\,\delta_{l'_{1\bar{1}},l_{1\bar{1}}}\delta_{l'_{2\bar{2}},l_{2\bar{2}}} \\
&\equiv B_p\,\delta_{l'_{1\bar{2}},l_{1\bar{2}}}\delta_{l'_{2\bar{1}},l_{2\bar{1}}}\delta_{l'_{\bar{1}\bar{2}},l_{\bar{1}\bar{2}}}\delta_{l'_{1\bar{1}},l_{1\bar{1}}}\delta_{l'_{2\bar{2}},l_{2\bar{2}}}
\end{aligned}
$$

Next special case: $\langle l'_{12} = 0 | \{l_{ij}\} \rangle$

$$= \frac{1}{l_{12}!} \langle l'_{12} = 0 | (k_+^{12})^{l_{12}} | l_{12} = 0 \rangle$$

$$= \vdots$$

$$\vdots$$

$$= A_1^{\prime(1)} \dots A_1^{\prime(l_{12})} \langle l'_{12} = 0, l'_{1\bar{2}} - l_{12}, l'_{2\bar{1}} - l_{12}, l'_{\bar{1}\bar{2}} + l_{12} | l_{12} = 0 \rangle$$

where,

$$A_1^{\prime(i)} = -\frac{l'_{\bar{1}\bar{2}} + i}{l_{12} + i - 1}.$$

Next, the most general case: $\langle\{l'_{ij}\}|\{l_{ij}\}\rangle$

$$\begin{aligned} & \langle\{l'_{ij}\}|\{l_{ij}\}\rangle = \langle l'_{12}-1|k_-^{12}|\{l_{ij}\}\rangle \\ \equiv\ & A_0^{(1)}\langle l'_{12}-1|l_{12}-1\rangle + A_1^{(1)}\langle l'_{12}-1|l_{1\bar{2}}-1, l_{2\bar{1}}-1, l_{\bar{1}\bar{2}}+1\rangle \end{aligned}$$

After p^{th} iteration, for example if $p = l'_{12}$

$$\sum_{q=0}^{p}\left[\sum_{\{s_i\}_q}{}' \left(A_{s_1}^{(1)}A_{s_2}^{(2)}\dots A_{s_p}^{(p)}\right)\langle l'_{12}=0|l_{12}-p+q, l_{1\bar{2}}-q, l_{2\bar{1}}-q, l_{\bar{1}\bar{2}}+q\rangle\right]$$

with $\{s_i\}_q \equiv P\left(\underbrace{1,1,\dots,1}_{q \text{ times}},\underbrace{0,0,\dots,0}_{p-q \text{ times}}\right)$

Normalization of local loop states: Iterative normalization

$\langle\{l'_{ij}\}|\{l_{ij}\}\rangle =$

$$\sum_{q=0}^{l'_{12}}\left[\left[\sum_{\{s_i\}_q}{}^{'}\left(A^{(1)}_{s_1}A^{(2)}_{s_2}\ldots A^{(l'_{12})}_{s_{l'_{12}}}\right)\right]\frac{(-1)^{l_{12}-l'_{12}+q}(l'_{\bar{1}\bar{2}}+l_{12}-l'_{12}+q)!}{l'_{\bar{1}\bar{2}}!(l_{12}-l'_{12}+q)!}\tilde{B}^q_{l'_{12}}\right]$$

$$\delta_{l'_{1\bar{2}}+l'_{12},l_{1\bar{2}}+l_{12}}\delta_{l'_{2\bar{1}}+l'_{12},l_{2\bar{1}}+l_{12}}\delta_{l'_{\bar{1}\bar{2}}-l'_{12},l_{\bar{1}\bar{2}}-l_{12}}\delta_{l'_{1\bar{1}},l_{1\bar{1}}}\delta_{l'_{2\bar{2}},l_{2\bar{2}}}$$

$$\tilde{A}'^{\,i}_q = -\frac{l'_{\bar{1}\bar{2}}+i}{l_{12}-l'_{12}+q+i-1} \quad ; \quad \tilde{B}^q_{l_{12}} = \frac{(l_{1\bar{2}}+l_{2\bar{1}}+l_{\bar{1}\bar{2}}+l_{1\bar{1}}+l_{2\bar{2}}+1-q)!}{(l_{1\bar{2}}-q)!\,(l_{2\bar{1}}+l_{\bar{1}\bar{2}}+l_{1\bar{1}}+l_{2\bar{2}}+1)!}\times\frac{(l_{2\bar{1}}+l_{\bar{1}\bar{2}}+l_{1\bar{1}}+l_{2\bar{2}}+1)!}{(l_{2\bar{1}}-q)!\,(l_{\bar{1}\bar{2}}+l_{1\bar{1}}+l_{2\bar{2}}+1+q)!}$$

$$\times\frac{(l_{\bar{1}\bar{2}}+l_{1\bar{1}}+l_{2\bar{2}}+1+q)!}{(l_{\bar{1}\bar{2}}+q)!\,(l_{1\bar{1}}+l_{2\bar{2}}+1)!}\times(l_{1\bar{1}}+1)\,(l_{2\bar{2}}+1)$$

Strong Coupling Perturbation Expansion

The strong coupling vacuum satisfying $H_e|0\rangle = 0$

The state with no loop present
Unperturbed vacuum energy $E_0^{(0)} = 0$.

Rayleigh-Schrödinger perturbation theory gives the corrections to the vacuum energy as:

$$E_0 = E_0^{(0)} + \frac{1}{g^2}E_0^{(1)} + \frac{1}{g^4}E_0^{(2)} + \frac{1}{g^6}E_0^{(3)} + \frac{1}{g^8}E_0^{(4)} + \ldots\ldots$$

Corrections to $E_0^{(0)}$

First order:

$$\langle 0|H_I|0\rangle = 0; \quad H_I = H_{mag}$$

$\Rightarrow$ All odd orders of corrections to vacuum energy do vanish

Only Even order corrections do contribute

Leading order correction:

$$E_0^{(2)} = \sum_{n_1 \neq 0} \frac{\langle 0|H_I|n_1\rangle\langle n_1|H_I|0\rangle}{\langle n_1|n_1\rangle \left(E_0 - E_0^{n_1}\right)} = \sum_{n_1 \neq 0} \frac{|\langle n_1|H_I|0\rangle|^2}{\langle n_1|n_1\rangle \left(E_0 - E_0^{n_1}\right)}$$

- For a N-plaquettes latice, N no. of $|n_1\rangle$ states created by a single action of $\mathrm{Tr}U_{plaquette}$ on $|0\rangle$.
- For the single plaquette states $E_0^{n_1} = 4 \times \frac{3}{4} = 3$.

$$E_0^{(2)} = N\frac{|\langle L(\tilde{x}) = 1|2\mathrm{Tr}U_{plaquette}|0\rangle|^2}{\langle L(\tilde{x}) = 1|L(\tilde{x}) = 1\rangle} \times \frac{1}{\left(-4 \times \frac{3}{4}\right)} = N \times \frac{2^2}{-3} \qquad (17)$$

This correction matches exactly to current literature.

To check the viability further:

Next to leading order correction $E_0^{(4)}$

$$\sum_{\{n_i\}_{\neq 0}} \frac{\langle 0|H_I|n_1\rangle\langle n_1|H_I|n_2\rangle\langle n_2|H_I|n_3\rangle\langle n_3|H_I|0\rangle}{\langle n_1|n_1\rangle\langle n_2|n_2\rangle\langle n_3|n_3\rangle\left(E_0-E_0^{n_1}\right)\left(E_0-E_0^{n_2}\right)\left(E_0-E_0^{n_3}\right)}$$
$$-E_0^{(2)}\sum_{\{n_1\}_{\neq 0}} \frac{\langle 0|H_I|n_1\rangle\langle n_1|H_I|0\rangle}{\langle n_1|n_1\rangle\left(E_0-E_0^{n_1}\right)^2}$$

- $|n_1\rangle$ as well as the $|n_3\rangle$ are the single plaquette states, located anywhere on the lattice.
- Another intermediate state $|n_2\rangle$ which is a two plaquette state.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2} = 8 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = 6$.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2} = 8 \times \frac{1}{2}\left(\frac{1}{2}+1\right) = 6$.
- Two completely overlapping plaquettes $|n_2\rangle \equiv |L(\tilde{x}) = 2\rangle$ with $E_0^{n_2} = 4 \times \frac{2}{2}\left(\frac{2}{2}+1\right) = 8$.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2 2} = 8 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = 6$.
- Two completely overlapping plaquettes $|n_2\rangle \equiv |L(\tilde{x}) = 2\rangle$ with $E_0^{n_2 2} = 4 \times \frac{2}{2}\left(\frac{2}{2} + 1\right) = 8$.
- Two separate plaquettes with overlap along any of the link: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1(\pm e_2)) = 1\rangle$, four possibilities, with $E_0^{n_2 2} = \frac{2}{2}\left(\frac{2}{2} + 1\right) + 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{13}{2}$.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2} = 8 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = 6$.
- Two completely overlapping plaquettes $|n_2\rangle \equiv |L(\tilde{x}) = 2\rangle$ with $E_0^{n_2} = 4 \times \frac{2}{2}\left(\frac{2}{2} + 1\right) = 8$.
- Two separate plaquettes with overlap along any of the link: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1(\pm e_2)) = 1\rangle$, four possibilities, with $E_0^{n_2} = \frac{2}{2}\left(\frac{2}{2} + 1\right) + 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{13}{2}$.
- Two plaquettes are touching each other at one of its four corners, i.e $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1 \pm e_2) = 1\rangle$, four possibilities with $E_0^{n_2} = 6$.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2} = 8 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = 6$.
- Two completely overlapping plaquettes $|n_2\rangle \equiv |L(\tilde{x}) = 2\rangle$ with $E_0^{n_2} = 4 \times \frac{2}{2}\left(\frac{2}{2} + 1\right) = 8$.
- Two separate plaquettes with overlap along any of the link: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1(\pm e_2)) = 1\rangle$, four possibilities, with $E_0^{n_2} = \frac{2}{2}\left(\frac{2}{2} + 1\right) + 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{13}{2}$.
- Two plaquettes are touching each other at one of its four corners, i.e $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1 \pm e_2) = 1\rangle$, four possibilities with $E_0^{n_2} = 6$.
- By the action of type (b) terms in the Hamiltonian, the two plaquette state with verical extension of two lattice units and horizontal extension of one, i.e $|n_2\rangle = H_{3/5}|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_2) = 1, N_1(\tilde{x} \pm \frac{e_2}{2}) = 1\rangle$ two possibilities with $E_0^{n_2} = 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{9}{2}$.

Possibility for the two plaquette states:

- Two disjoint plaquettes qith no overlap: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_2) = 1\rangle$. $N - 9$ possibilities with $E_0^{n_2} = 8 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = 6$.
- Two completely overlapping plaquettes $|n_2\rangle \equiv |L(\tilde{x}) = 2\rangle$ with $E_0^{n_2} = 4 \times \frac{2}{2}\left(\frac{2}{2} + 1\right) = 8$.
- Two separate plaquettes with overlap along any of the link: $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1(\pm e_2)) = 1\rangle$, four possibilities, with $E_0^{n_2} = \frac{2}{2}\left(\frac{2}{2} + 1\right) + 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{13}{2}$.
- Two plaquettes are touching each other at one of its four corners, i.e $|n_2\rangle = H_1|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1 \pm e_2) = 1\rangle$, four possibilities with $E_0^{n_2} = 6$.
- By the action of type (b) terms in the Hamiltonian, the two plaquette state with verical extension of two lattice units and horizontal extension of one, i.e $|n_2\rangle = H_{3/5}|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_2) = 1, N_1(\tilde{x} \pm \frac{e_2}{2}) = 1\rangle$ two possibilities with $E_0^{n_2} = 6 \times \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{9}{2}$.
- Similarly, two plaquette state with verical extension of one lattice units and horizontal extension of two, i.e $|n_2\rangle = H_{2/4}|n_1\rangle \equiv |L(\tilde{x}_1) = 1, L(\tilde{x}_1 \pm e_1) = 1, N_2(\tilde{x} \pm \frac{e_1}{2}) = 1\rangle$ with $E_0^{n_2} = \frac{9}{2}$

4th order correction to vacuum energy:

Explicit calculation incorporating all the coefficients for the Hamiltonian actions and the norm of each state calculated yields,

$$E_0^{(4)} = N\frac{2 \times 163}{3^4 \times 13} \equiv N \times 2^4 \times \frac{163}{8424} \quad (18)$$

At this order also the result matches exactly (i.e upto 12th decimal place) with the present literature

Proves the viability of our formulation.
In the same way the strong coupling perturbation correction to any loop state can be performed within this scheme and note that this scheme is independent of any cluster size or lattice size.

- Prepotential formulation: Local loop formulation.
- Local Loop operators and states: The diagrametic techniques, a set of 'lattice Feynman rules' to perform all computations diagrammatically bypassing long and tedious algebraic calculations.
- Introducing Fusion variables: Most suitable for Hamiltonian Dynamics.
- Enumeration of loop states by fixing integer valued quantum numbers throughout the lattice.
- Physical loops: Non-intersecting, true for any dimension.
- Complete loop dynamics is represented by fusion quantum numbers.
- Completely geometric method without using complicated Clebsch-Gordon coefficients and hence generalization to higher SU(N) group is possible.

Reference:

R. Anishetty and I. Raychowdhury, Phys. Rev. D **90**, no. 11, 114503 (2014).

THANK YOU