Phase Transition in Gross Neveu model with Borici Creutz fermions

Jishnu Goswami

IIT Kanpur

Collaborators:
D. Chakrabarti, IIT Kanpur
S. Basak, NISER Bhubaneswar

Ref: J.Goswami,D.Chakrabarti and S.Basak PhysRevD.91.014507(arxiv 1409.7999)

Perspectives and Challenges in Lattice Gauge Theory
TIFR, Mumbai

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Outline

Introduction

- Simulation with dynamical fermions in a lattice is always a challenging task.
- The famous no-go theorem:Lattice fermion actions with,
  - locality
  - chiral symmetry
  - hermiticity

  must produce massless fermions in multiples of two in continuum limit.
- There exist lot of fermion prescriptions to avoid fermion doubling caused by the naive fermions.
- Every model has its own advantges and also individual shortcomings.

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- Lattice fermions and shortcomings
  - Wilson fermion: No chiral symmetry
  - Staggered fermion: Doublers not remove totally and rooting needed
  - Domain wall and Overlap fermion: Complicated simulation algorithms
- Another possible way is lattice action with 2 massless species, the minimum number required by the no-go theorem, called minimal-doubling fermions.
- There are three types of minimally doubled actions,
  - Karsten-Wilczek
  - Borici-Creutz
  - Twisted-ordering types.
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Classification:KW,TO and BC

- Naive fermions in momentum space

$$aD_{fm}(p) = i\gamma_\mu \sin p_\mu a$$

16 massless fermions in continuum limit (for ap 0 and π) known as doublers,

- Wilson type fermions are,

$$aD_{fm}(p) = i\gamma_\mu \sin p_\mu a + (1 - \cos(ap_\mu))$$

So now the 15 out of 16 fermions get large mass (O($1/a$)) and decoupled in continum (only for ap=0 remains)

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- Now if we further modify it by adding a gamma matrix with the second term,

$$aD_{fm}(p) = i\gamma_\mu \sin p_\mu a + i\gamma_4 M_f(p_\mu)$$

where, $M_f(p_\mu) = (1 - \cos(ap_\mu))$ is flavored mass term. [[1]]
Karsten-Wilczek fermions.
Has only two zeros,$(0, 0, 0, 0)$ and (π, π, π, π)

- Now this type of term preserves the chiral symmetry but breaks the hypercubic symmetry.

[1] M.Creutz et. al arxiv :1011.0761

Classification:KW,TO and BC

- Another type is twisted ordering , Lets start by writing instead of a single gamma matrix sum over all gamma matrices,

$$aD_{fm}(p) = i\gamma_\mu \sin p_\mu a - i\gamma_\mu M_f(p_\mu)$$

[2] M.Creutz et. al PRD 82,074502 (2010)

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$$D(p) = \underbrace{(\sin p_1 + \cos p_1 - 1)i\gamma_1 + (\sin p_2 + \cos p_2 - 1)i\gamma_2}_{\text{4 zeros for every p at } 0 \text{ and } \frac{\pi}{2}}$$

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- After twisting p_1 and p_2 of $\cos$ terms,[2]

$$D(p) = \underbrace{(\sin p_1 + \cos p_2 - 1)i\gamma_1 + (\sin p_2 + \cos p_1 - 1)i\gamma_2}_{\text{2 zeros at } (0,0) \text{ and } (\frac{\pi}{2}, \frac{\pi}{2})}$$

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- So twisting the order of gamma matrices in the second term reduces the zeros, Similarly in 4D after twist we get only two zeros,

$$aD_{fm}(p) = \underbrace{i\gamma_\mu \sin p_\mu a + i\gamma_{\mu-1} M_f(p_\mu)}_{\text{2 zeros at (0,0,0,0) and } (\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})}$$

$$\mu - 1 = \begin{cases} 1,2,3, & \text{if } \mu\text{=2,3,4} \\ 4 & \text{if } \mu\text{=1} \end{cases}$$

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Figure : Four zeros In 2D

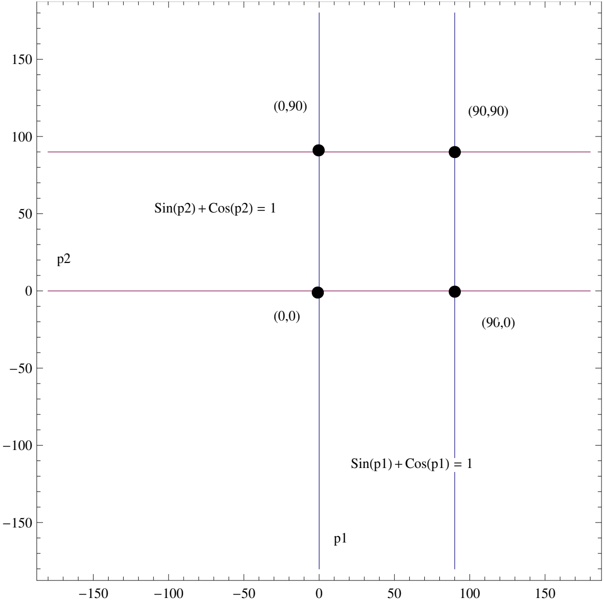
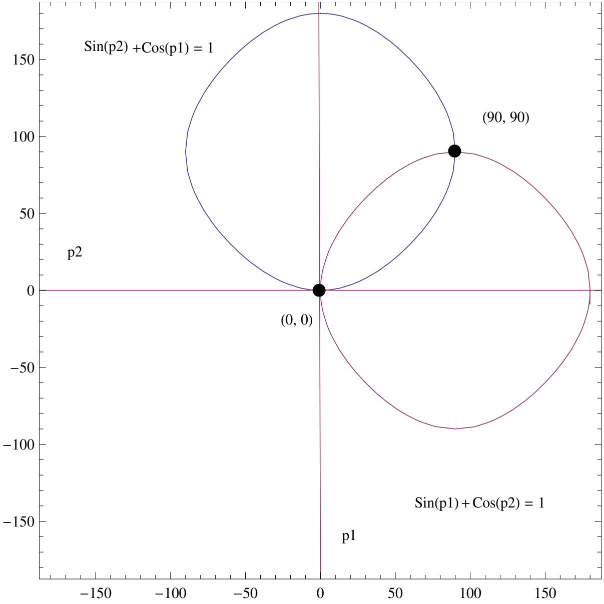


Figure : Two zeros In 2D after twisting



Borici-Creutz fermion

- Now if instead of twisting we take a different set of gamma matrices with the $\cos$ term,
  Then the Dirac operator is $(a = 1)$,[3]

$$D_{BC}(p) = \underbrace{\sum_{\mu} \left[i\gamma_\mu \sin p_\mu - i(\gamma'_\mu)(1 - cos(p_\mu) \right]}_{\text{Two zeros at (0,0,0,0) and } (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})}.$$

Important relations:,

$$\gamma'_\mu = \sum_{\mu} \gamma_\mu \Gamma \gamma_\mu$$

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \; and \; \Gamma^2 = 1$$

$$\{\Gamma, \gamma_\mu\} = \{\Gamma, \gamma'_\mu\} = 1.$$

[3] M. Creutz JHEP 0804,017(2008),A. Borici PRD 78 017(2010)

More on BC fermion

- Hypercubic symmetry is broken so we can introduce other dimension counter terms but as long as $M_f(p_\mu)$ is cubic symmetric, only three and four dimensional counterterms will be required.
- For BC action we here only analyse the dimension three counter term(c_3) and tune the coefficient of 4 dimension counter terms to zero,
- So its look like ($a = 1$)

$$D_{BC}(p) = \sum_{\mu} \left[i\gamma_\mu \sin p_\mu + i(\Gamma - \gamma_\mu) cos(p_\mu) \right] + i(c_3 - 2)\Gamma$$

- Now the term c_3 changes the number and postion of the zeros,

$$c_3 = \begin{cases} 0 & \text{two zeros (0,0,0,0) and } (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) \\ 4 & \text{two zeros } (\pi, \pi, \pi, \pi) \text{ and } (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) \\ 2 & \text{no zeros} \end{cases}$$

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- Borici Creutz action in 4 dimensional space is written as,

$$S_{BC} = \sum_n \Big[\frac{1}{2}\sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) - \frac{ir}{2}\sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu)(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) + ic_3 \bar{\psi}_n \Gamma \psi_n + m\bar{\psi}_n \psi_n\Big]$$

- We write this action using the hopping and onsite operators as,

$$S_{BC} = \sum_n \Big[\sum_\mu (\bar{\psi}_n P_\mu^+ \psi_{n+\mu} - \bar{\psi}_n P_\mu^- \psi_{n-\mu}) + \bar{\psi}_n \hat{M} \psi_n\Big]$$

where the hopping operators are defined as $P_\mu^+ = \frac{\gamma_\mu}{2}(1 - ir) + \frac{ir\Gamma}{2}$, $P_\mu^- = \frac{\gamma_\mu}{2}(1 + ir) - \frac{ir\Gamma}{2}$ and the onsite operator $\hat{M} = m + i(c_3 - 2r)\Gamma$.

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- In the strong coupling limit the effective action is,

$$S_{eff} = \sum_n \Big[\sum_\mu \mathsf{T}r(M(n)(P_\mu^+)^T M(n+\hat\mu)(P_\mu^-)^T) + \mathsf{T}r(\hat{M}M(n)) - \mathrm{Tr}(\log M(n)) \Big]$$

where $M(n) = \bar\psi(n)\psi(n)/N_c$ and the trace is over spinor indices.

- The condensate(VEV) of M(n), has both σ and π_Γ condensates,

$$\langle M(n)\rangle = M_0 = \sigma I_4 + i\Gamma\pi_\Gamma.$$

- After putting this into previous equation we get the effective action as,

$$S_{eff} = N_c \Big[4\sigma^2(1+r^2) + 2{\pi_\Gamma}^2(1+r^2) + 4m\sigma - 4(c_3 - 2r)\pi_\Gamma - 2\log(\sigma^2 + {\pi_\Gamma}^2) \Big]$$

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Phase Diagram in 4D

Gap equations

- From the saddle point solutions the gap equations are,

$$2\sigma(1+r^2) + m - \frac{\sigma}{\sigma^2 + {\pi_\Gamma}^2} = 0,$$
$$\pi_\Gamma(1+r^2) - (c_3 - 2r) - \frac{\pi_\Gamma}{\sigma^2 + {\pi_\Gamma}^2} = 0.$$

- These equantions can be solved analytically for $m = 0$. Setting $\sigma \to 0$, we get the chiral boundaries for massless Borici-Creutz fermions at,

$$c_3 - 2r = \pm\sqrt{\frac{1+r^2}{2}}.$$

- For $r = 1$ the chiral boundaries are at $\bar{c_3} = c_3 - 2 = \pm 1$.
- We get two solutions for the condensates for $m = 0$ and $r = 1$ as

$$\sigma = 0, \pi_\Gamma = \frac{1}{4}(\bar{c_3} \pm \sqrt{8 + \bar{c_3}^2}\ ).$$

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$$and\ \sigma = \frac{\sqrt{1-\bar{c_3}^2}}{2}, \pi_\Gamma = -\frac{\bar{c_3}}{2}.$$

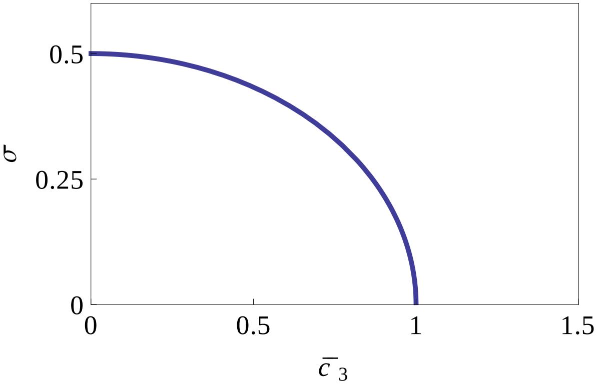

Figure : $\bar{c_3}$ vs σ for Borici-Creutz fermions when m=0 and r=1.

[4]

[4] T. Misumi, JHEP **1208**, 068 (2012)[Similar things done for KW fermions]

Gross Neveu Model in 2 dimensions

Multipicity of the free Dirac operator

- The Borici-Creutz action has already been defined previously. In 2D, $\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$, $\{\Gamma, \gamma_\mu\} = 1$, and $\Gamma^2 = \frac{1}{2}$.[(2×2) gamma matrices]
- The free Dirac operator in momentum space is written as,

$$D_{BC}(p) = \sum_\mu [i\gamma_\mu \sin p_\mu + i(\Gamma - \gamma_\mu)cos(p_\mu)] + i(c_3 - 2)\Gamma.$$

  - For $c_3 = 0$ and $c_3 = 4$ only one zero of the Dirac operator but dispersion becomes unphysical.
  - For $0 < c_3 < 0.59$ and $3.41 < c_3 < 4$ the Dirac operator has only two zeros i.e this is the region of minimal doubling.
  - And for the rest of the region i.e $0.59 < c_3 < 3.41$ the Dirac operator has four zeros. Out of those zeros, we get correct continuum limit of the Dirac operator only when $p_1 = p_2$.

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Four fermi interaction

- The free action (with $r = 1$) is

$$S_{BC} = \sum_n \Big[\frac{1}{2}\sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) - \frac{i}{2}\sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu)(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) + i(c_3 - 2)\bar{\psi}_n \Gamma \psi_n + m\bar{\psi}_n \psi_n\Big]$$

- After including the four fermi interactions,

$$S_{BCGN} = \sum_n \Big[S_{BC} - \frac{g^2}{2N}[(\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n i\Gamma \psi_n)^2]\Big].$$

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Four fermi interaction

- To linearize the four fermion interactions, we introduce two real auxiliary fields σ and π_Γ:

$$\begin{aligned}\sigma(n) &= m - \frac{g^2}{N}(\bar{\psi}_n \psi_n) \\ \pi_\Gamma(n) &= c_3 - 2 - \frac{g^2}{N}(\bar{\psi}_n i\Gamma \psi_n).\end{aligned}$$

- The effective action becomes,

$$\begin{aligned}\tilde{S}_{eff} &= N\Big[\frac{1}{2g^2}[(\sigma - m)^2 + (\pi_\Gamma - c_3 + 2)^2 \\ &- \int \frac{d^2k}{(2\pi)^2}\log\left[\sigma^2 + \frac{\pi_\Gamma^2}{2} + \pi_\Gamma(C + D) + C^2 + D^2\right]\Big]\end{aligned}$$

- Then the gap equations are obtained as

$$\begin{aligned}\frac{(\sigma - m)}{g^2} &= \int \frac{d^2k}{(2\pi)^2} \frac{2\sigma}{(\sigma^2 + \frac{\pi_\Gamma^2}{2} + \pi_\Gamma(C+D) + C^2 + D^2)} , \\ \frac{(\pi_\Gamma - c_3 + 2)}{g^2} &= \int \frac{d^2k}{(2\pi)^2} \frac{\pi_\Gamma + (C+D)}{(\sigma^2 + \frac{\pi_\Gamma^2}{2} + \pi_\Gamma(C+D) + C^2 + D^2)} ;\end{aligned}$$

where,

$$\begin{aligned} C &= \sin(k_1) - \frac{1}{2}(\cos(k_1) - \cos(k_2)), \\ D &= \sin(k_2) - \frac{1}{2}(\cos(k_2) - \cos(k_1)). \end{aligned}$$

- For exact chiral structure m=0 and at the chiral boundary σ=0,
- the mass of σ is zero on the critical line which indicates a second order phase transition.

$$m_\sigma \propto V \frac{\delta^2 S_{eff}^{-}}{\delta \sigma^2}\bigg|_{(c_3)_c} = 0$$

Phase Diagram in parameter space

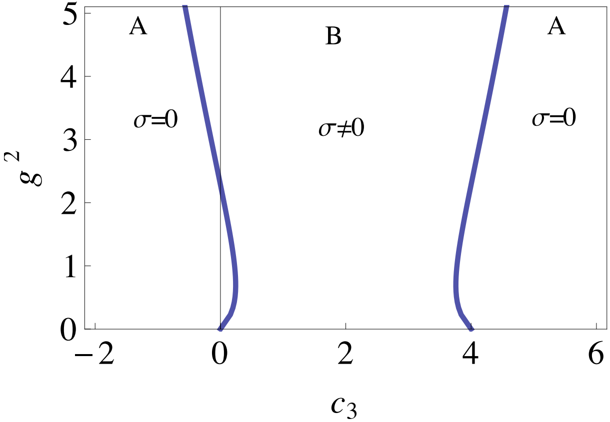

Figure : Chiral boundaries in the parametric space i.e. c_3 vs g^2 for BC fermions

HMC of the model

- For numerical simulation we take $c_3 = 0 + \epsilon$ where $\epsilon = 10^{-5}$
- The lattice version of the action is written as,

$$S = \bar{\psi}_i M_{ij} \psi_j + \frac{N}{2g^2}(\sigma^2 + \pi_\Gamma^2),$$

$$M_{ij} = D_{ij} + \frac{1}{4} \sum_{<x,\tilde{x}>} (\sigma + i\pi_\Gamma \Gamma).$$

- where the auxiliary fields are defined in the dual lattice sites $\tilde{x}$ surrounding the direct lattice site [5] x.

[5] S. J. Hands, A. Kocić, J. B. Kogut, Nucl. Phys. B**390**, 355 (1993), Ann. Phys. **224**, 29 (1993).

HMC of the Model

- Where D_{ij} is the BC Dirac operator:

$$D_{ij} = \frac{1}{2}\gamma_\mu(\delta_{j,i+\mu} - \delta_{j,i-\mu}) + \frac{i}{2}(\Gamma - \gamma_\mu)(\delta_{j,i+\mu} + \delta_{j,i-\mu}) \\ -((2-\epsilon)i\Gamma - m)\delta_{i,j}.$$

- We take $(M^\dagger M)$ to make it real and positive definite and integrate out the fermion fields.
- With psedofermions and $N_f = 2N = 4$ the action becomes

$$S = \phi^\dagger (M^\dagger M)^{-1}\phi + \frac{1}{g^2}(\sigma^2 + \pi_\Gamma^2).$$

- We take the mass values as 0.01,0.02 and 0.03.

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HMC of the model

- We simulate our model by hybrid monte carlo (HMC) method and evaluate the order parameter for the chiral phase transition $\langle\sigma\rangle$ as a function of coupling constant. We use point sources to estimate the condensate.

$$
\begin{aligned}
\langle\bar{\psi}\psi\rangle &= -\langle TrM^{-1}\rangle \\
\langle\sigma\rangle &= -\beta\langle\bar{\psi}\psi\rangle \\
\text{where } \beta &= \frac{1}{g^2}.
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- The configurations are generated by considering stepsize ($\triangle t$=0.1) in the leapfrog method and ten steps per trajectory in the molecular dynamics chain.
First 500 ensembles are rejected for thermalization and data are collected for next 16000 ensembles.

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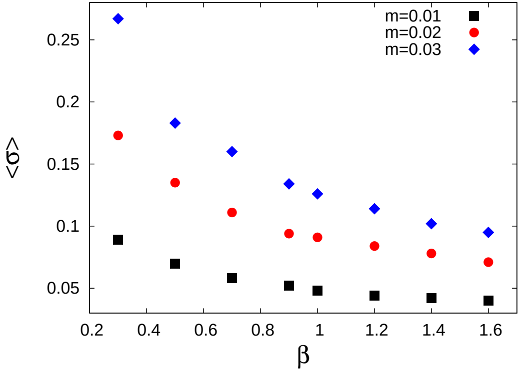

Figure : $\langle\sigma\rangle$ vs β of m=0.01,0.02 & 0.03 for Gross-Neveu model with BC fermions in a 32×32 lattice

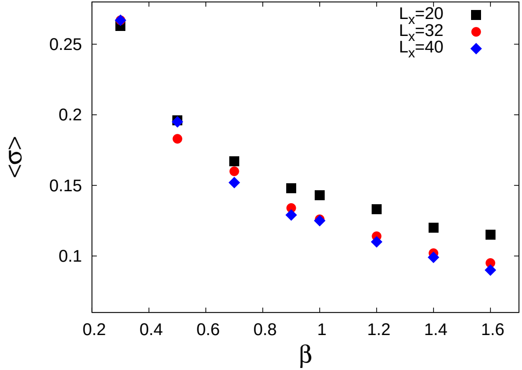

Figure : Finite volume effects of $\langle\sigma\rangle$ vs β for m=0.03 of three different lattice sizes 20×20, 32×32, and 40×40

Mass Spectrum

- Next we find the mass spectrum of GN model using this fermion formulation,
- we present some preliminary results of the mass spectrum by calculating the following correlators,

$$
\begin{aligned}
c1(x,t) &= \bar{\psi}(x,t)\gamma_5\psi(x,t) \\
c2(x,t) &= \bar{\psi}(x+n,t)\gamma_5\Big[\psi(x+n,t)+\psi(x-n,t)\Big] \\
c3(x,t) &= \Big[\psi(x+n,t)-\psi(x-n,t)\Big]\gamma_5\Big[\psi(x+n,t)-\psi(x-n,t)\Big]
\end{aligned}
$$

Where, $n = 2$

- then calculate the effective mass using,

$$m_{eff} = \log\frac{ci(t)}{ci(t+1)}$$

Where, $i = 1, 2, 3$

Preliminary Results

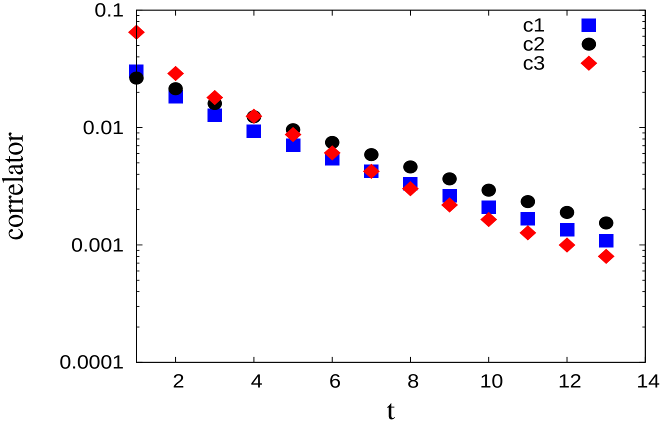

Figure : corr vs t for m=0.03 and $L = 16 \times 48$

Preliminary Results

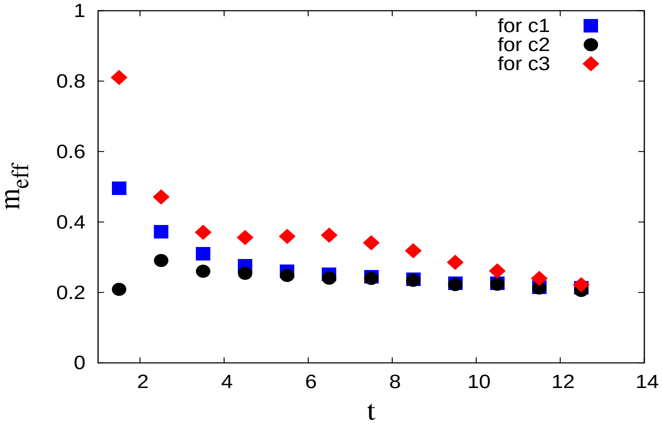

Figure : m_{eff} vs t for m=0.03 and $L = 16 \times 48$

Summary

- We have studied the Gross-Neveu model with minimally doubled fermion action which has been proposed by Creutz and Borici.
- We have analytically shown a second order phase transition boundary from symmetric to broken chiral phase.
- Then we have studied the model with HMC algorithm. The order parameter $\langle\sigma\rangle$ is plotted against $\beta = 1/g^2$ shows chiral phase transition.
- We have calculate the mass spectrum of GN model and present some preliminary results for that,
- Issues(4 D),
  Counter terms ?? Renormaization ?? operator mixing issues ??

Thank you