THEORY

# Lattice Gauge Theory with Equivariant Gauge Fixing

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# Chiral Gauge Theory & the Problem of Rough Gauge Fields

THEORY

Nielsen-Ninomiya Theorem / Chiral Anomalies / Ginsparg-Wilson

→ Explicit chiral symmetry breaking by lattice fermions

**Lattice Chiral Gauge Theories break gauge invariance.**

One way (*Luescher 1999, 2000*):
Modify chiral symmetry on lattice in accordance with Ginsparg-Wilson relation

$$\hat{\gamma}_5(U) = \gamma_5(1 - aD_{\mathrm{GW}}(U))$$

*U*-dependent fermion measure —> Solve integrability condition on the space of *U*

- Exact Solution was found for the Abelian case
- For the non-Abelian case, a solution was found only in Perturbation Theory (requiring an infinite number of irrelevant counter-terms), no non-perturbative solution is known.

# Rough Gauge Fields

Saha Institute of Nuclear Physics, Calcutta — Advancement of Knowledge

THEORY

Consider $S = S_{\rm GI} + S_{\rm NI}(U)$ $\qquad U_{x\mu} \to g_x U_{x\mu}\, g^\dagger_{x+\mu}\,, \quad g_x \in G$

For example: $S_{\rm GI}$ may be the plaquette action, and

$$S_{\rm NI}(U) \sim -\sum (U_{x\mu} + U^\dagger_{x\mu}) \sim \int_x A^2_\mu(x)$$

$$Z = \int \mathcal{D}U \exp(-S) \quad = \int \mathcal{D}U \exp(-S_{GI} - S_{NI}(U_{x\mu}))$$

$$= \int \mathcal{D}U \exp(-S_{GI} - S_{NI}(g_x U_{x\mu} g_{x+\mu}))$$

$$= \int \mathcal{D}U \mathcal{D}\phi \exp(-S_{GI} - S_{NI}(\phi^\dagger_x U_{x\mu} \phi_{x+\mu}))$$

After integrating both sides by gauge degrees of freedom with $\phi^\dagger_x = g_x$ and $\int \mathcal{D}\phi = 1$

The $\phi$ fields are radially frozen scalar fields, they are random because any point on the gauge orbit is as likely as any other point, making the gauge fields effectively rough

# The Reduced Model

THEORY

The new action $S(\phi_x^\dagger U_{x\mu}\phi_{x+\mu})$

is now gauge-invariant under the transformations:

$$U_{x\mu} \to h_x U_{x\mu}\, h_{x+\mu}^\dagger, \quad \phi_x \to h_x \phi_x, \quad h_x \in G$$

Sometimes the original action without the scalars (longitudinal gauge degrees of freedom) is described as one in the *vector picture*, and the action with the scalars as one in the *Higgs picture*:

$$S_{\mathrm{V}}(U_{x\mu}) = S_{\mathrm{H}}(U_{x\mu};\, \phi_x)|_{\phi=1}$$

The *reduced model* is defined by the action on the trivial orbit: $U_{x\mu} = g_x\, 1\, g_{x+\mu}^\dagger$

In the Higgs picture, the reduced model is obtained by setting $U_{x\mu} = 1$

- In the current example, the reduced model is a pure scalar theory
- If the non-invariant part is due to chiral fermions, the reduced model would be a fermion-scalar theory

# Lattice Chiral Gauge Theories that broke chiral symmetry

THEORY

- *Wilson-Yukawa / Smit-Swift* model
- *Domain-wall waveguide* model
- *Eichten-Preskill* model

They were investigated in the reduced model, and in all cases fermion doublers (dynamically generated through strong interactions between fermions and scalars) made a back-door entry to make the theory vector-like.

Because of strong interactions between the fermions and the scalars (*lgdofs*), neutral fermions were formed:

Supposing, the original theory is written with a charged *L*-handed and a neutral *R*-handed fermion

$$\Psi_{\rm L}(x) \to h_x \Psi_{\rm L}(x), \quad \Psi_{\rm R}(x) \to \Psi_{\rm R}(x)$$

the undoubled spectrum contains only neutral free Dirac fermion:

$$\Psi^{(\rm n)}(x) = \phi_x^\dagger \Psi_{\rm L}(x) + \Psi_{\rm R}(x)$$

where the composite field $\phi^\dagger \Psi_{\rm L}$ is a neutral *L*-handed field because $\phi_x \to h_x \phi_x$

## How to decouple the scalars (longitudinal gauge degrees of freedom)?

THEORY

Gauge fix: to remove the redundant degrees of freedom

The standard *Fadeev-Popov* procedure is restricted to continuum perturbation theory

$\det \partial_\mu \mathcal{D}_\mu$ is real, but is not guaranteed to be positive in general

$|\det \partial_\mu \mathcal{D}_\mu| = |\det \Box|\, |\det(1 - g\Box^{-1}C)|$ The operator $C$ is linear in $A_\mu$

where $\mathcal{D}^{ac}_\mu = \delta^{ac}\partial_\mu + \mathrm{ad}(A_\mu)^{ac}, \quad \mathrm{ad}(A_\mu)^{ac} = -f^{abc}A^b_\mu$

$|\det \Box|$ being independent of the gauge field gets cancelled in the functional integral between the numerator and the denominator

In Perturbation Theory, $\det(1 - g\Box^{-1}C) > 0$ Hence $|\det \partial_\mu \mathcal{D}_\mu| = \det \partial_\mu \mathcal{D}_\mu$

and it can be exponentiated in terms of ghost fields (Grassmann)

THEORY

The gauge-fixed theory:

$$Z = \int \mathcal{D}A_\mu \;\det(\partial_\mu \mathcal{D}_\mu)\, \delta(F(A_\mu))\, \exp(-S_{\rm GI})$$

The linearity of the gauge fixing function, for example $F^a(A_\mu) = \partial_\mu A^a_\mu - f^a(x)$ allows to replace the delta function by an exponential

$$\delta(F(A_\mu)) \to \exp\left(-\int d^4x\, \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right)$$

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm GI} + \bar\eta(\partial_\mu \mathcal{D}_\mu)\eta + \frac{1}{2\xi}(F(A_\mu))^2$$

Introducing an auxiliary field $B^a = \frac{i}{\xi}\partial_\mu A^a_\mu$ (commuting and scalar)

the gauge fixing part can also be written as:

$$= \bar\eta(\partial_\mu \mathcal{D}_\mu)\eta + \frac{\xi}{2}(B^a)^2 - iB^a F^a(A_\mu)$$

$$= \bar\eta(\partial_\mu \mathcal{D}_\mu)\eta + \frac{\xi}{2}(B^a)^2 - iB^a \partial_\mu A^a_a$$

The theory in this form is BRST-invariant

# BRST and Conventions

THEORY

Infinitesimal gauge transformation $\equiv \delta_\theta$

BRST variation of a physical field is obtained by replacing the local parameter by the ghost field $\eta$ : $\delta_{\rm B} \equiv \delta_{\theta\to\eta}$

$$\delta_{\rm B} A_\mu = \mathcal{D}_\mu \eta \qquad \delta_{\rm B}\Psi = -i\eta\Psi \qquad \delta_{\rm B}\bar{\Psi} = i\eta^a\bar{\Psi}T^a$$

$\delta_{\rm B}$ is nilpotent and anti commutes with Grassmann

$\delta_{\rm B}^2\,\Psi = 0 \;\Rightarrow\; \delta_{\rm B}\eta = -i\eta^2$ or, equivalently, $\delta_{\rm B}\eta^c = \frac{1}{2} f^{abc}\eta^a\eta^b$

Then impose: $\delta_{\rm B}\bar{\eta} = -iB \quad \text{and} \quad \delta_{\rm B} B = 0$

- Can one construct a local, gauge-fixed, BRST-invariant lattice formulation of Yang-Mills theory?
- This will help construct a lattice (non-pert.) formulation of Chiral Gauge Theories
- A BRST-invariant gauge-fixed lattice theory may possibly contain new phases in addition to the familiar confinement phase.

# The Gribov Problem

*Gribov 1978, Singer 1978*

THEORY

Gauge invariant observables remains intact provided the following integral

$$\int \mathcal{D}g\mathcal{D}\eta\mathcal{D}\bar{\eta}\mathcal{D}B\,\exp(-S_{\rm GF}[A^g_\mu,\eta,\bar{\eta},B])$$

over an orbit is a non-zero constant.

In Perturbation Theory, which is a saddle-point approximation around the classical vacuum $A_\mu = 0$ , this condition is satisfied.

Not necessarily true non-perturbatively, because of Gribov copies, i.e., multiple solutions of the gauge fixing condition, $\partial_\mu A^g_\mu = 0$ on the same orbit.

In this situation, the correct condition would be $\sum \mathrm{sign}\,(\det\,\partial_\mu \mathcal{D}_\mu) \neq 0$

*Hirschfeld, 1979*

To test the condition in a well-defined non-perturbative set-up, one needs to do it on the lattice regulator.

$\delta_{\rm B} A^a_\mu = \mathcal{D}^{ab}_\mu \eta^b \quad \Rightarrow \quad \delta_{\rm B}\, U_{x\mu} = i\,(U_{x\mu}\eta_{x+\mu} - \eta_x U_{x\mu})$ :Lattice BRST transformation

# A No-Go Theorem and BRST with Compact Gauge Fields

*Neuberger 1987*

THEORY

Start with a manifestly gauge invariant lattice gauge theory, the Wilson way:

$$Z = \int \mathcal{D}U \exp(-S_{\mathrm{GI}}[U])$$

One then inserts, *a la* Fadeev-Popov: $Z_{\mathrm{GF}} = \int \mathcal{D}g\, \mathcal{D}\eta\, \mathcal{D}\bar{\eta}\, \mathcal{D}B \exp(-S_{\mathrm{GF}}[U^g, \eta, \bar{\eta}, B])$

$$S_{\mathrm{GF}} = \sum_x 2t\,[-i\,\mathrm{tr}(B\,F(U)) + \mathrm{tr}(\bar{\eta}\,\delta_B F(U))] + \sum_x \xi\, g^2\, \mathrm{tr}(B^2)$$

$$= \sum_x 2t\,[\delta_B \mathrm{tr}(\bar{\eta}\,F(U))] + \sum_x \xi\, g^2\, \mathrm{tr}(B^2)$$

$Z_{\mathrm{GF}}$ is required to be independent of $U$ , so that only a constant was inserted in $Z$

If this requirement is fulfilled, gauge-invariant correlation functions of the gauge-fixed theory are identical to those of the unfixed (manifestly gauge invariant) theory.

THEORY

Now $$\frac{Z_{\rm GF}}{dt} = \int \mathcal{D}g\,\mathcal{D}\eta\,\mathcal{D}\bar\eta\,\mathcal{D}B\;\delta_B\Big[\sum_x 2{\rm tr}(\bar\eta\,F(U))\Big]$$
$$= 0$$

Indeed $Z_{\rm GF}|_{t=1} = Z_{\rm GF}|_{t=0}$ independent of $U$

But $Z_{\rm GF}|_{t=0} = 0$

because the integrand is then devoid of any Grassmann variables

$$\Rightarrow Z_{\rm GF}|_{t=1} = Z_{\rm GF} = 0$$

Expectation value of any gauge-invariant operator has the indeterminate form $0/0$

Deeper reason for the zeros is likely to be the Gribov copies *Testa 1998*

If pure Yang Mills cannot be non-perturbatively gauge fixed, there would be no hope for non-Abelian Chiral Gauge Theories in the gauge-fixing approach.

# Equivariant BRST

*Schaden 1999*

THEORY

Gauge fixing on a coset space $SU(N)/U(1)^{N-1}$ *Golterman-Shamir 2005*

**BRST only on a part of the gauge group** $G$ **leaving unfixed a subgroup** $H \subset G$

Divide the generators into a sub-algebra $T^i$ generating the subgroup $H$

and the rest of them $T^\alpha$ spanning the coset space $G/H$

$$V_\mu = V_\mu^a T^a = A_\mu^i T^i + W_\mu^\alpha T^\alpha$$

Two elements of the subgroup $H$ should again be in $H$

$$f_{i\alpha j} = -f_{\alpha ij} = -f_{ij\alpha} = 0$$

Introduce $\mathcal{G}/\mathcal{H}$ valued ghost fields: $\eta = \eta^\alpha T^\alpha, \quad \bar{\eta} = \bar{\eta}^\alpha T^\alpha$ $\quad \mathrm{tr}(\eta T^i) = 0$

and the auxiliary field: $B = B^\alpha T^\alpha$

THEORY

All the ghost sector fields will transform under infinitesimal *H gauge* transformations in the same way as adjoint representation field:

$$\delta_\omega \eta = -i[\omega, \eta], \quad \omega \in \mathcal{H}$$

To check this, we must verify: $[\omega, \eta] \in \mathcal{G}/H$

Now $[\omega, \eta] = i\, \omega^i \eta^\alpha f^{i\alpha b} T^b$

So the question is where does the index $b$ belong?

We observe that $f^{i\alpha j} = -f^{ij\alpha} = 0$

Hence $[\omega, \eta] = i\, \omega^i \eta^\alpha f^{i\alpha\beta} T^\beta \quad \in \mathcal{G}/H$ as it should be

# eBRST Transformations

THEORY

eBRST transformation rules for the physical fields are the same as before

$$sV_\mu = \mathcal{D}_\mu(V)\eta$$

Now consider *G* and *G/H* parts separately:

$$sW_\mu = [\mathcal{D}_\mu(V)\eta]_{\mathcal{G}/H} = \partial_\mu\eta|_{\mathcal{G}/H} + i[V_\mu,\eta]_{\mathcal{G}/H} = \partial_\mu\eta|_{\mathcal{G}/H} + i[A_\mu,\eta]_{\mathcal{G}/H} + i[W_\mu,\eta]_{\mathcal{G}/H}$$

$$= \mathcal{D}_\mu(A)\eta + i[W_\mu,\eta]_{\mathcal{G}/H}$$

In the above, we have used (as shown before) $[A_\mu,\eta] \in \mathcal{G}/H \quad \Rightarrow \mathcal{D}_\mu(A)\eta \in \mathcal{G}/H$

$$sA_\mu = (\mathcal{D}_\mu(V)\eta)_{\mathcal{H}} = \partial_\mu\eta|_{\mathcal{H}} + i[V_\mu,\eta]_{\mathcal{H}} = i[A_\mu,\eta]_{\mathcal{H}} + i[W_\mu,\eta]_{\mathcal{H}}$$

$$= i[W_\mu,\eta]_{\mathcal{H}}$$

$[W_\mu,\eta]$ has, in general, components in both $\mathcal{G}/H$ and $\mathcal{H}$

for special cases like $G = SU(2), \quad [W_\mu,\eta] \in \mathcal{U}(1) = \mathcal{H}$

Project usual BRST rule for $\eta$ onto coset space $\mathcal{G}/H$

$$\text{Require } \mathrm{tr}(s\eta \mathrm{T}^\alpha) = \mathrm{tr}(\delta_\mathrm{B}\eta T^\alpha) \text{ for } T^\alpha \in \mathcal{G}/H$$
$$\text{and } \mathrm{tr}(s\eta T^i) = 0 \text{ for } T^i \in \mathcal{H}$$

$\Rightarrow \quad s\eta = (-i\eta^2)_{\mathcal{G}/H} \equiv -i\eta^2 + X$ where $X \equiv (i\eta^2)_{\mathcal{H}} = 2iT^j \mathrm{tr}(\eta^2 T^j)$

**This modification affects the nilpotency of eBRST**

$$s\ \mathrm{tr}(\eta^2 T^j) = \mathrm{tr}(s\eta^2 T^j) = \mathrm{tr}[\{(-\eta^2 + X)\eta - \eta(-i\eta^2 + X)\}T^j]$$

$$= \mathrm{tr}[(X\eta - \eta X)T^j] = X^i\eta^\alpha \mathrm{tr}([T^i, T^\alpha]T^j) = if^{i\alpha\beta}X^i\eta^\alpha \mathrm{tr}(T^\beta T^j) = 0$$

Hence $\quad sX = 0$

$$s^2\eta = s[-i\eta^2 + X] = -is\eta^2 + sX = -i[-i\eta^2 + X]\eta + i\eta[-i\eta^2 + X]$$

$$= -iX\eta + i\eta X = -i[X, \eta] = \delta_X \eta$$

It does not vanish, and equals an infinitesimal gauge transformation in ***H*** with (commuting) parameter ***X***

We take the usual BRST rule for the $\bar{\eta}$ $s\bar{\eta} = -iB$

We require $s^2\bar{\eta} = \delta_X \bar{\eta}$

That gives: $-isB = -i[X, \bar{\eta}] \;\Rightarrow\; sB = [X, \bar{\eta}] \neq 0$

$$s^2 B = s[X\bar{\eta} - \bar{\eta}X] = X(s\bar{\eta}) - (s\bar{\eta})X = X(-iB) - (-iB)X \;= -i[X, B] = \delta_X B$$

The second eBRST variation of any physical field follows from the fact that the standard BRST transformation is nilpotent, so that only the ***X*** part in ***sC*** leads to a non-vanishing result,

$$s^2 A_\mu = \mathcal{D}_\mu(A)X = \delta_X A_\mu$$

$$s^2 W_\mu = -i[X, W_\mu] = \delta_X W_\mu$$

For $\psi$ in fundamental representation, $s\psi = -i\eta\psi$

$$s^2\psi = -i(s\eta)\psi + i\eta(s\psi) = -[-i\eta^2 + X]\psi + i\eta[-i\eta\psi] = -iX\psi$$

$s^2$ is equivariantly nilpotent

$s^2$ is H gauge transformation with a commuting parameter $X$

eBRST is still nilpotent on any ***H*** invariant operator

# eBRST-invariant Action Density

THEORY

$\xi g^2 \mathrm{tr}(B^2)$ term of the standard BRST invariant action is no longer eBRST invariant

The action now will have the generic form

$$S_{eBRST} = s\mathcal{W} \;\; \text{where} \;\; \mathcal{W} \;\; \text{is} \;\; \mathcal{H} \;\; \text{invariant}$$

$$sS_{eBRST} = s^2\mathcal{W} = \delta_X \mathcal{W} = 0$$

$$S_{eBRST} = s\,\mathrm{tr}(2\bar{\eta}\mathcal{F} + i\xi g^2 \bar{\eta} B)$$

The $H$ covariant gauge condition is : $\mathcal{F}(V) = \mathcal{D}_\mu(A) W_\mu \equiv \partial_\mu W_\mu + i[A_\mu, W_\mu]$

Clearly $\mathcal{F} \in \mathcal{G}/H$

and $\mathcal{F}$ has the same $H$ transformations as the ghost sector fields $\delta_\omega \mathcal{F} = -i[\omega, \mathcal{F}]$

$\mathcal{W}$ is indeed $H$ invariant

However, this is not the most general eBRST action

THEORY

But, one can already see how **the No-Go by Neuberger is evaded**

The new term gives rise to, in addition to the $B^2$ term, a 4 ghost term, characteristic of eBRST actions

It turns out that still $\frac{d}{dt} Z_{eBRST} = 0$ But $Z_{eBRST}|_{t=1} = Z_{eBRST}|_{t=0} \neq 0$

The formalism can be extended to include anti-eBRST and the so-called ghost flip symmetry

*Baulieu & Thierry-Mieg 1982*

The most general action density: $S_{s,\bar{s}} = -s\bar{s}\,\mathrm{tr}(W^2 + \xi g^2 \bar{\eta}\eta) = \bar{s}s\,\mathrm{tr}(W^2 + \xi g^2 \bar{\eta}\eta)$

This action has eBRST, anti-eBRST, ghost flip, and H-gauge invariance

If the coset structure constants $f_{\alpha\beta\gamma}$ are all equal to zero, there is no difference between the two actions.

This happens for $G = SU(2), \quad H = U(1)$

# Rest of the Program for Chiral Gauge Theory or otherwise

THEORY

- The continuum action constructed so far has to be transcribed on to lattice. This can be done keeping all the symmetries except continuous Euclidean rotation
- The action still has ***H*** gauge invariance. This now needs to be gauge fixed, again in a way so that the No-Go by Neuberger is evaded
- The Abelian gauge fixing proposal *(Golterman-Shamir 1997)* has been tested reasonably well both in perturbation theory and in numerical simulations by *Bock, Golterman & Shamir* and by *Basak & De*
- However, the abelian proposal can also be taken as an alternate non-perturbative definition of the pure gauge theory (NEXT TALK), just as the non-abelian proposal can be taken as a non-perturbative BRST method for defining the pure Yang-Mills theory. Work is under progress for a numerical survey of the phase diagram.