

THEORY

Lattice Gauge Theory with Equivariant Gauge Fixing

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Chiral Gauge Theory & the Problem of Rough Gauge Fields



THEORY

Nielsen-Ninomiya Theorem / Chiral Anomalies / Ginsparg-Wilson

→ Explicit chiral symmetry breaking by lattice fermions

Lattice Chiral Gauge Theories break gauge invariance.

One way (*Luescher 1999, 2000*):

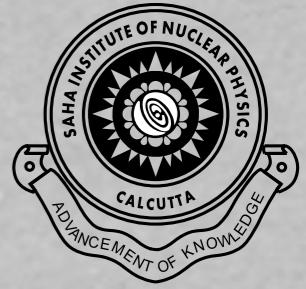
Modify chiral symmetry on lattice in accordance with Ginsparg-Wilson relation

$$\hat{\gamma}_5(U) = \gamma_5(1 - \textcolor{blue}{a} D_{\text{GW}}(U))$$

U -dependent fermion measure —> Solve integrability condition on the space of U

- Exact Solution was found for the Abelian case
- For the non-Abelian case, a solution was found only in Perturbation Theory (requiring an infinite number of irrelevant counter-terms), no non-perturbative solution is known.

Rough Gauge Fields



THEORY

Consider

$$S = S_{GI} + S_{NI}(U) \quad U_{x\mu} \rightarrow g_x U_{x\mu} g_{x+\mu}^\dagger, \quad g_x \in G$$

For example: S_{GI} may be the plaquette action, and

$$S_{NI}(U) \sim - \sum (U_{x\mu} + U_{x\mu}^\dagger) \sim \int_x A_\mu^2(x)$$

$$\begin{aligned} Z &= \int \mathcal{D}U \exp(-S) = \int \mathcal{D}U \exp(-S_{GI} - S_{NI}(U_{x\mu})) \\ &= \int \mathcal{D}U \exp(-S_{GI} - S_{NI}(g_x U_{x\mu} g_{x+\mu})) \\ &= \int \mathcal{D}U \mathcal{D}\phi \exp(-S_{GI} - S_{NI}(\phi_x^\dagger U_{x\mu} \phi_{x+\mu})) \end{aligned}$$

After integrating both sides by gauge degrees of freedom with $\phi_x^\dagger = g_x$ and $\int \mathcal{D}\phi = 1$

The ϕ fields are radially frozen scalar fields, they are random because any point on the gauge orbit is as likely as any other point, making the gauge fields effectively rough

The Reduced Model



THEORY

The new action $S(\phi_x^\dagger U_{x\mu} \phi_{x+\mu})$

is now gauge-invariant under the transformations:

$$U_{x\mu} \rightarrow h_x U_{x\mu} h_{x+\mu}^\dagger, \quad \phi_x \rightarrow h_x \phi_x, \quad h_x \in G$$

Sometimes the original action without the scalars (longitudinal gauge degrees of freedom) is described as one in the *vector picture*, and the action with the scalars as one in the *Higgs picture*:

$$S_V(U_{x\mu}) = S_H(U_{x\mu}; \phi_x)|_{\phi=1}$$

The *reduced model* is defined by the action on the trivial orbit: $U_{x\mu} = g_x \mathbf{1} g_{x+\mu}^\dagger$

In the Higgs picture, the reduced model is obtained by setting $U_{x\mu} = \mathbf{1}$

- In the current example, the reduced model is a pure scalar theory
- If the non-invariant part is due to chiral fermions, the reduced model would be a fermion-scalar theory



THEORY

Lattice Chiral Gauge Theories that broke chiral symmetry

- *Wilson-Yukawa / Smit-Swift* model
- *Domain-wall waveguide* model
- *Eichten-Preskill* model

They were investigated in the reduced model, and in all cases fermion doublers (dynamically generated through strong interactions between fermions and scalars) made a back-door entry to make the theory vector-like.

Because of strong interactions between the fermions and the scalars (*lgdofs*), neutral fermions were formed:

Supposing, the original theory is written with a charged *L*-handed and a neutral *R*-handed fermion

$$\Psi_L(x) \rightarrow h_x \Psi_L(x), \quad \Psi_R(x) \rightarrow \Psi_R(x)$$

the undoubled spectrum contains only neutral free Dirac fermion:

$$\Psi^{(n)}(x) = \phi_x^\dagger \Psi_L(x) + \Psi_R(x)$$

where the composite field $\phi^\dagger \Psi_L$ is a neutral *L*-handed field because $\phi_x \rightarrow h_x \phi_x$



THEORY

How to decouple the scalars (longitudinal gauge degrees of freedom)?

Gauge fix: to remove the redundant degrees of freedom

The standard *Faddeev-Popov* procedure is restricted to continuum perturbation theory

$\det \partial_\mu \mathcal{D}_\mu$ is real, but is not guaranteed to be positive in general

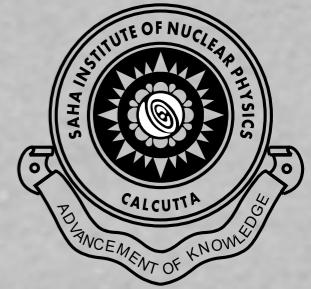
$|\det \partial_\mu \mathcal{D}_\mu| = |\det \square| |\det(1 - g\square^{-1}C)|$ The operator C is linear in A_μ

where $\mathcal{D}_\mu^{ac} = \delta^{ac} \partial_\mu + \text{ad}(A_\mu)^{ac}$, $\text{ad}(A_\mu)^{ac} = -f^{abc} A_\mu^b$

$|\det \square|$ being independent of the gauge field gets cancelled in the functional integral between the numerator and the denominator

In Perturbation Theory, $\det(1 - g\square^{-1}C) > 0$ Hence $|\det \partial_\mu \mathcal{D}_\mu| = \det \partial_\mu \mathcal{D}_\mu$

and it can be exponentiated in terms of ghost fields (Grassmann)



THEORY

The gauge-fixed theory:

$$Z = \int \mathcal{D}A_\mu \det(\partial_\mu \mathcal{D}_\mu) \delta(F(A_\mu)) \exp(-S_{\text{GI}})$$

The linearity of the gauge fixing function, for example $F^a(A_\mu) = \partial_\mu A_\mu^a - f^a(x)$ allows to replace the delta function by an exponential

$$\delta(F(A_\mu)) \rightarrow \exp\left(-\int d^4x \frac{1}{2\xi} (\partial_\mu A_\mu)^2\right)$$

→ $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{GI}} + \bar{\eta}(\partial_\mu \mathcal{D}_\mu)\eta + \frac{1}{2\xi} (F(A_\mu))^2$

Introducing an auxiliary field $B^a = \frac{i}{\xi} \partial_\mu A_\mu^a$ (commuting and scalar)

the gauge fixing part can also be written as:

$$\begin{aligned} &= \bar{\eta}(\partial_\mu \mathcal{D}_\mu)\eta + \frac{\xi}{2} (B^a)^2 - iB^a F^a(A_\mu) \\ &= \bar{\eta}(\partial_\mu \mathcal{D}_\mu)\eta + \frac{\xi}{2} (B^a)^2 - iB^a \partial_\mu A_\mu^a \end{aligned}$$

The theory in this form is BRST-invariant

BRST and Conventions



THEORY

Infinitesimal gauge transformation $\equiv \delta_\theta$

BRST variation of a physical field is obtained by replacing the local parameter by the ghost field η : $\delta_B \equiv \delta_{\theta \rightarrow \eta}$

$$\delta_B A_\mu = \mathcal{D}_\mu \eta \quad \delta_B \Psi = -i\eta \Psi \quad \delta_B \bar{\Psi} = i\eta^a \bar{\Psi} T^a$$

δ_B is nilpotent and anti commutes with Grassmann

$$\delta_B^2 \Psi = 0 \Rightarrow \delta_B \eta = -i\eta^2 \quad \text{or, equivalently,} \quad \delta_B \eta^c = \frac{1}{2} f^{abc} \eta^a \eta^b$$

Then impose: $\delta_B \bar{\eta} = -iB$ and $\delta_B B = 0$

- Can one construct a local, gauge-fixed, BRST-invariant lattice formulation of Yang-Mills theory?
- This will help construct a lattice (non-pert.) formulation of Chiral Gauge Theories
- A BRST-invariant gauge-fixed lattice theory may possibly contain new phases in addition to the familiar confinement phase.



The Gribov Problem

Gribov 1978, Singer 1978

THEORY

Gauge invariant observables remains intact provided the following integral

$$\int \mathcal{D}g \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}B \exp(-S_{\text{GF}}[A_\mu^g, \eta, \bar{\eta}, B])$$

over an orbit is a non-zero constant.

In Perturbation Theory, which is a saddle-point approximation around the classical vacuum $A_\mu = 0$, this condition is satisfied.

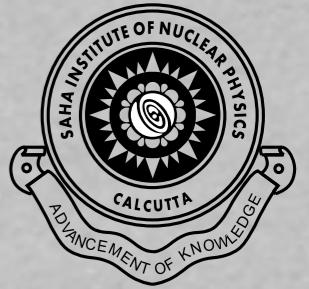
Not necessarily true non-perturbatively, because of Gribov copies, i.e., multiple solutions of the gauge fixing condition, $\partial_\mu A_\mu^g = 0$ on the same orbit.

In this situation, the correct condition would be $\sum \text{sign}(\det \partial_\mu \mathcal{D}_\mu) \neq 0$

Hirschfeld, 1979

To test the condition in a well-defined non-perturbative set-up, one needs to do it on the lattice regulator.

$$\delta_B A_\mu^a = \mathcal{D}_\mu^{ab} \eta^b \Rightarrow \delta_B U_{x\mu} = i (U_{x\mu} \eta_{x+\mu} - \eta_x U_{x\mu}) \quad \text{:Lattice BRST transformation}$$



THEORY

A No-Go Theorem and BRST with Compact Gauge Fields

Neuberger 1987

Start with a manifestly gauge invariant lattice gauge theory, the Wilson way:

$$Z = \int \mathcal{D}U \exp(-S_{\text{GI}}[U])$$

One then inserts, *a la* Fadeev-Popov: $Z_{\text{GF}} = \int \mathcal{D}g \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}B \exp(-S_{\text{GF}}[U^g, \eta, \bar{\eta}, B])$

$$\begin{aligned} S_{\text{GF}} &= \sum_x 2t [-i \text{tr}(B F(U)) + \text{tr}(\bar{\eta} \delta_B F(U))] + \sum_x \xi g^2 \text{tr}(B^2) \\ &= \sum_x 2t [\delta_B \text{tr}(\bar{\eta} F(U))] + \sum_x \xi g^2 \text{tr}(B^2) \end{aligned}$$

Z_{GF} is required to be independent of U , so that only a constant was inserted in Z

If this requirement is fulfilled, gauge-invariant correlation functions of the gauge-fixed theory are identical to those of the unfixed (manifestly gauge invariant) theory.

Now

$$\begin{aligned}\frac{Z_{\text{GF}}}{dt} &= \int \mathcal{D}g \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}B \delta_B \left[\sum_x 2\text{tr}(\bar{\eta} F(U)) \right] \\ &= 0\end{aligned}$$

Indeed $Z_{\text{GF}}|_{t=1} = Z_{\text{GF}}|_{t=0}$ independent of U

But $Z_{\text{GF}}|_{t=0} = 0$

because the integrand is then devoid of any Grassmann variables

$$\Rightarrow Z_{\text{GF}}|_{t=1} = Z_{\text{GF}} = 0$$

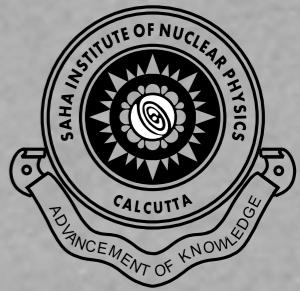
Expectation value of any gauge-invariant operator has the indeterminate form 0/0

Deeper reason for the zeros is likely to be the Gribov copies *Testa 1998*

If pure Yang Mills cannot be non-perturbatively gauge fixed, there would be no hope for non-Abelian Chiral Gauge Theories in the gauge-fixing approach.

Equivariant BRST

Schaden 1999



THEORY

Gauge fixing on a coset space $SU(N)/U(1)^{N-1}$

Golterman-Shamir 2005

BRST only on a part of the gauge group G leaving unfixed a subgroup $H \subset G$

Divide the generators into a sub-algebra T^i generating the subgroup H

and the rest of them T^α spanning the coset space G/H

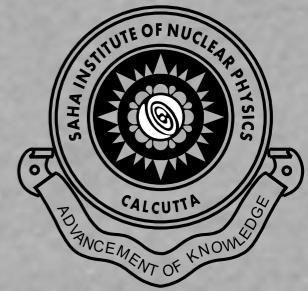
$$V_\mu = V_\mu^a T^a = A_\mu^i T^i + W_\mu^\alpha T^\alpha$$

Two elements of the subgroup H should again be in H

$$f_{i\alpha j} = -f_{\alpha ij} = -f_{ij\alpha} = 0$$

Introduce \mathcal{G}/\mathcal{H} valued ghost fields: $\eta = \eta^\alpha T^\alpha$, $\bar{\eta} = \bar{\eta}^\alpha T^\alpha$ $\text{tr}(\eta T^i) = 0$

and the auxiliary field: $B = B^\alpha T^\alpha$



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All the ghost sector fields will transform under infinitesimal *H* gauge transformations in the same way as adjoint representation field:

$$\delta_\omega \eta = -i[\omega, \eta], \quad \omega \in \mathcal{H}$$

To check this, we must verify: $[\omega, \eta] \in \mathcal{G}/H$

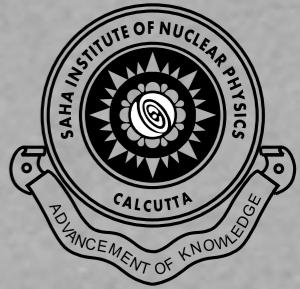
Now $[\omega, \eta] = i\omega^i\eta^\alpha f^{i\alpha b}T^b$

So the question is where does the index b belong?

We observe that $f^{i\alpha j} = -f^{ij\alpha} = 0$

Hence $[\omega, \eta] = i\omega^i\eta^\alpha f^{i\alpha\beta}T^\beta \in \mathcal{G}/H$ as it should be

eBRST Transformations



eBRST transformation rules for the physical fields are the same as before

$$sV_\mu = \mathcal{D}_\mu(V)\eta$$

Now consider G and G/H parts separately:

$$\begin{aligned} sW_\mu &= [\mathcal{D}_\mu(V)\eta]_{G/H} = \partial_\mu\eta|_{G/H} + i[V_\mu, \eta]_{G/H} = \partial_\mu\eta|_{G/H} + i[A_\mu, \eta]_{G/H} + i[W_\mu, \eta]_{G/H} \\ &= \mathcal{D}_\mu(A)\eta + i[W_\mu, \eta]_{G/H} \end{aligned}$$

In the above, we have used (as shown before) $[A_\mu, \eta] \in \mathcal{G}/H \Rightarrow \mathcal{D}_\mu(A)\eta \in \mathcal{G}/H$

$$\begin{aligned} sA_\mu &= (\mathcal{D}_\mu(V)\eta)_{\mathcal{H}} = \partial_\mu\eta|_{\mathcal{H}} + i[V_\mu, \eta]_{\mathcal{H}} = i[A_\mu, \eta]_{\mathcal{H}} + i[W_\mu, \eta]_{\mathcal{H}} \\ &= i[W_\mu, \eta]_{\mathcal{H}} \end{aligned}$$

$[W_\mu, \eta]$ has, in general, components in both \mathcal{G}/H and \mathcal{H}

for special cases like $G = SU(2)$, $[W_\mu, \eta] \in \mathcal{U}(1) = \mathcal{H}$

Project usual BRST rule for η onto coset space \mathcal{G}/H

Require $\text{tr}(s\eta T^\alpha) = \text{tr}(\delta_B \eta T^\alpha)$ for $T^\alpha \in \mathcal{G}/H$
 and $\text{tr}(s\eta T^i) = 0$ for $T^i \in \mathcal{H}$

$$\Rightarrow s\eta = (-i\eta^2)_{\mathcal{G}/H} \equiv -i\eta^2 + X \quad \text{where} \quad X \equiv (i\eta^2)_{\mathcal{H}} = 2iT^j \text{tr}(\eta^2 T^j)$$

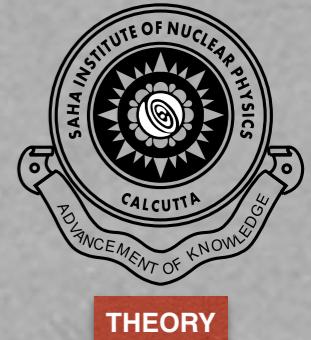
This modification affects the nilpotency of eBRST

$$\begin{aligned} s \text{tr}(\eta^2 T^j) &= \text{tr}(s\eta^2 T^j) = \text{tr}[\{(-\eta^2 + X)\eta - \eta(-i\eta^2 + X)\}T^j] \\ &= \text{tr}[(X\eta - \eta X)T^j] = X^i \eta^\alpha \text{tr}([T^i, T^\alpha]T^j) = if^{i\alpha\beta} X^i \eta^\alpha \text{tr}(T^\beta T^j) = 0 \end{aligned}$$

Hence $sX = 0$

$$\begin{aligned} s^2\eta &= s[-i\eta^2 + X] = -is\eta^2 + sX = -i[-i\eta^2 + X]\eta + i\eta[-i\eta^2 + X] \\ &= -iX\eta + i\eta X = -i[X, \eta] = \delta_X \eta \end{aligned}$$

It does not vanish, and equals an infinitesimal gauge transformation in H with (commuting) parameter X



We take the usual BRST rule for the $\bar{\eta}$ $s\bar{\eta} = -iB$

We require $s^2\bar{\eta} = \delta_X\bar{\eta}$

That gives: $-isB = -i[X, \bar{\eta}] \Rightarrow sB = [X, \bar{\eta}] \neq 0$

$$s^2B = s[X\bar{\eta} - \bar{\eta}X] = X(s\bar{\eta}) - (s\bar{\eta})X = X(-iB) - (-iB)X = -i[X, B] = \delta_X B$$

The second eBRST variation of any physical field follows from the fact that the standard BRST transformation is nilpotent, so that only the \mathbf{X} part in \mathbf{sc} leads to a non-vanishing result,

$$s^2A_\mu = \mathcal{D}_\mu(A)X = \delta_X A_\mu$$

$$s^2W_\mu = -i[X, W_\mu] = \delta_X W_\mu$$

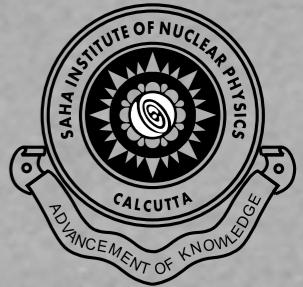
For ψ in fundamental representation, $s\psi = -i\eta\psi$

$$s^2\psi = -i(s\eta)\psi + i\eta(s\psi) = -[-i\eta^2 + X]\psi + i\eta[-i\eta\psi] = -iX\psi$$

s^2 is equivariantly nilpotent

s^2 is H gauge transformation with a commuting parameter X

eBRST is still nilpotent on any \mathbf{H} invariant operator



eBRST-invariant Action Density

$\xi g^2 \text{tr}(B^2)$ term of the standard BRST invariant action is no longer eBRST invariant

The action now will have the generic form

$$S_{eBRST} = s\mathcal{W} \text{ where } \mathcal{W} \text{ is } \mathcal{H} \text{ invariant}$$

$$sS_{eBRST} = s^2\mathcal{W} = \delta_X \mathcal{W} = 0$$

$$S_{eBRST} = s \text{ tr}(2\bar{\eta}\mathcal{F} + i\xi g^2 \bar{\eta}B)$$

The H covariant gauge condition is : $\mathcal{F}(V) = \mathcal{D}_\mu(A)W_\mu \equiv \partial_\mu W_\mu + i[A_\mu, W_\mu]$

Clearly $\mathcal{F} \in \mathcal{G}/H$

and \mathcal{F} has the same H transformations as the ghost sector fields $\delta_\omega \mathcal{F} = -i[\omega, \mathcal{F}]$

\mathcal{W} is indeed H invariant



THEORY

However, this is not the most general eBRST action

But, one can already see how **the No-Go by Neuberger is evaded**

The new term gives rise to, in addition to the B^2 term, a 4 ghost term, characteristic of eBRST actions

It turns out that still $\frac{d}{dt} Z_{eBRST} = 0$ But $Z_{eBRST}|_{t=1} = Z_{eBRST}|_{t=0} \neq 0$

The formalism can be extended to include anti-eBRST and the so-called ghost flip symmetry

Baulieu & Thierry-Mieg 1982

The most general action density: $S_{s,\bar{s}} = -s\bar{s} \operatorname{tr}(W^2 + \xi g^2 \bar{\eta}\eta) = \bar{s}s \operatorname{tr}(W^2 + \xi g^2 \bar{\eta}\eta)$

This action has eBRST, anti-eBRST, ghost flip, and H-gauge invariance

If the coset structure constants $f_{\alpha\beta\gamma}$ are all equal to zero, there is no difference between the two actions.

This happens for $G = SU(2)$, $H = U(1)$

Rest of the Program for Chiral Gauge Theory or otherwise

- The continuum action constructed so far has to be transcribed on to lattice. This can be done keeping all the symmetries except continuous Euclidean rotation
- The action still has **H** gauge invariance. This now needs to be gauge fixed, again in a way so that the No-Go by Neuberger is evaded
- The Abelian gauge fixing proposal (*Golterman-Shamir 1997*) has been tested reasonably well both in perturbation theory and in numerical simulations by *Bock, Golterman & Shamir* and by *Basak & De*
- However, the abelian proposal can also be taken as an alternate non-perturbative definition of the pure gauge theory (NEXT TALK), just as the non-abelian proposal can be taken as a non-perturbative BRST method for defining the pure Yang-Mills theory. Work is under progress for a numerical survey of the phase diagram.