SU(2) loop states and Hydrogen atoms.

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Work done with Manu Mathur.

PERSPECTIVES AND CHALLENGES IN LATTICE GAUGE THEORY
20 February, 2015

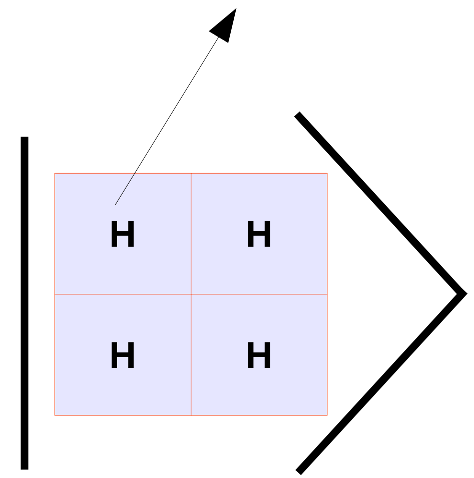
Plan of the talk and key results

PURE SU(2) GAUGE THEORY

A. KINEMATICS:

SU(2) LOOP HILBERT SPACE ≡

H: Hydrogen atom



NO MANDELSTAM CONSTRAINTS.
NO BIANCI IDENTITY CONSTRAINTS

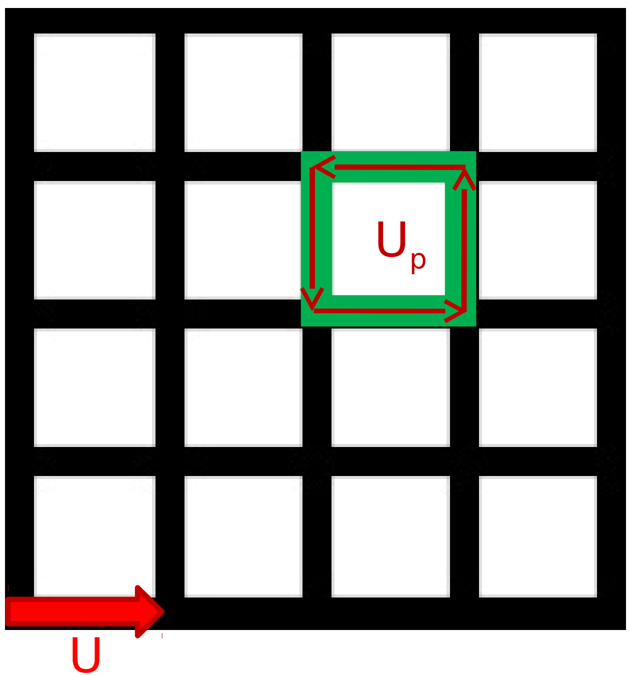
B. DYNAMICS:

- Nearest spin interaction Hamiltonian in the weak coupling limit.
- No local Gauss law [only a global one] .
- Dynamics described by Dynamical symmetry group of H atom [SO(4,2)].

Hamiltonian LGT * at a glance.

$$\mathbf{H} = \frac{\mathbf{g^2}}{\mathbf{2a}} \sum_{\mathbf{l}} \mathbf{E^2(l)} + \frac{\mathbf{4}}{\mathbf{ag^2}} \sum_{\mathbf{p}} \mathbf{Tr}\left[\mathbf{U_p} + \mathbf{U_p^\dagger}\right] \equiv \frac{\mathbf{g^2}}{\mathbf{2a}} \mathbf{H_E} + \frac{\mathbf{4}}{\mathbf{ag^2}} \mathbf{H_B}.$$

Where U is the link operator and E is the corresponding conjugate electric field.



Quantization

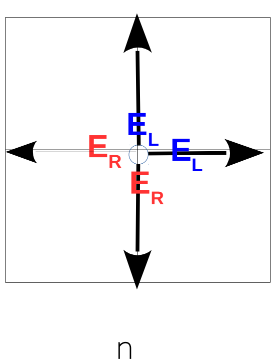
$$[\mathbf{E_L^\beta}, \mathbf{U_{ij}}] = -\tau^\beta_{\mathbf{ik}} \mathbf{U_{kj}}$$

$$[\mathbf{E_R^\beta}, \mathbf{U_{ij}}] = \mathbf{U_{ik}} \tau^\beta_{\mathbf{kj}}$$

$$[\mathbf{E_L^\alpha}, \mathbf{E_L^\beta}] = \mathbf{i f^{\alpha\beta\gamma} E_L^\gamma}$$

Gauss's Law constraint

At each site $\sum \mathbf{E|Phys\rangle = 0}$



* Kogut and Susskind (1975)

Motivation

3 Issues with Kogut Susskind formulation :

1. **Spurious gauge degrees of freedom** — Gauss law constraints

↓

Wilson loops — Highly non trivial Mandelstam constraints!

2. Strong coupling limit

Continuum limit.Physical region

$\frac{1}{g^2} = 0$ —————————————— $\frac{1}{g^2} \to \infty$

Naive weak coupling perturbation theory fails as H_B has continuous spectrum!

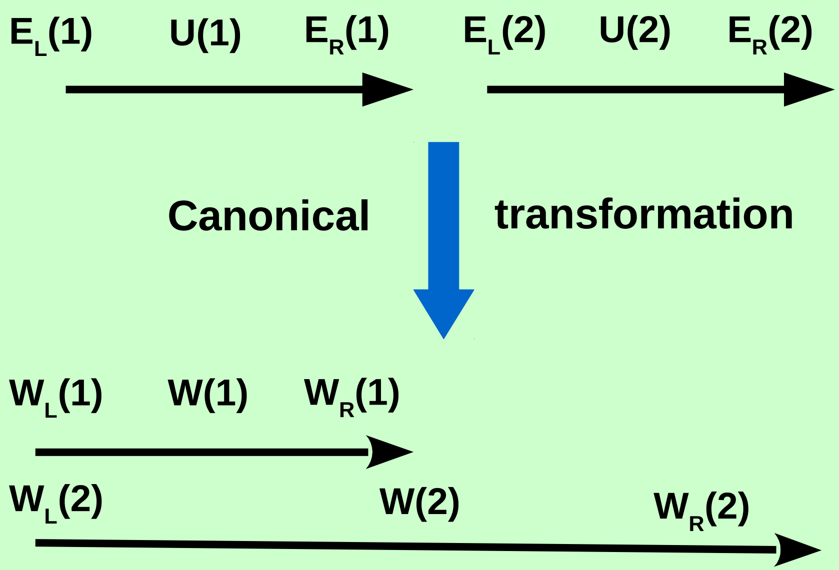
3.

Matrix element of H_B in the loop Hilbert space is highly complicated if you remove the loop redundancies.*

* Manu Mathur, Nuclear Physics B 779 (2007) 32–62

Canonical transformations.

- A basic canonical transformation looks like the following:



$$\mathbf{W}(\mathbf{1}) = \mathbf{U}(\mathbf{1}) \qquad \mathbf{W}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{1}) = \mathbf{E}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{1}) + \mathbf{E}^{\mathbf{a}}_{\mathbf{L}}(\mathbf{2})$$

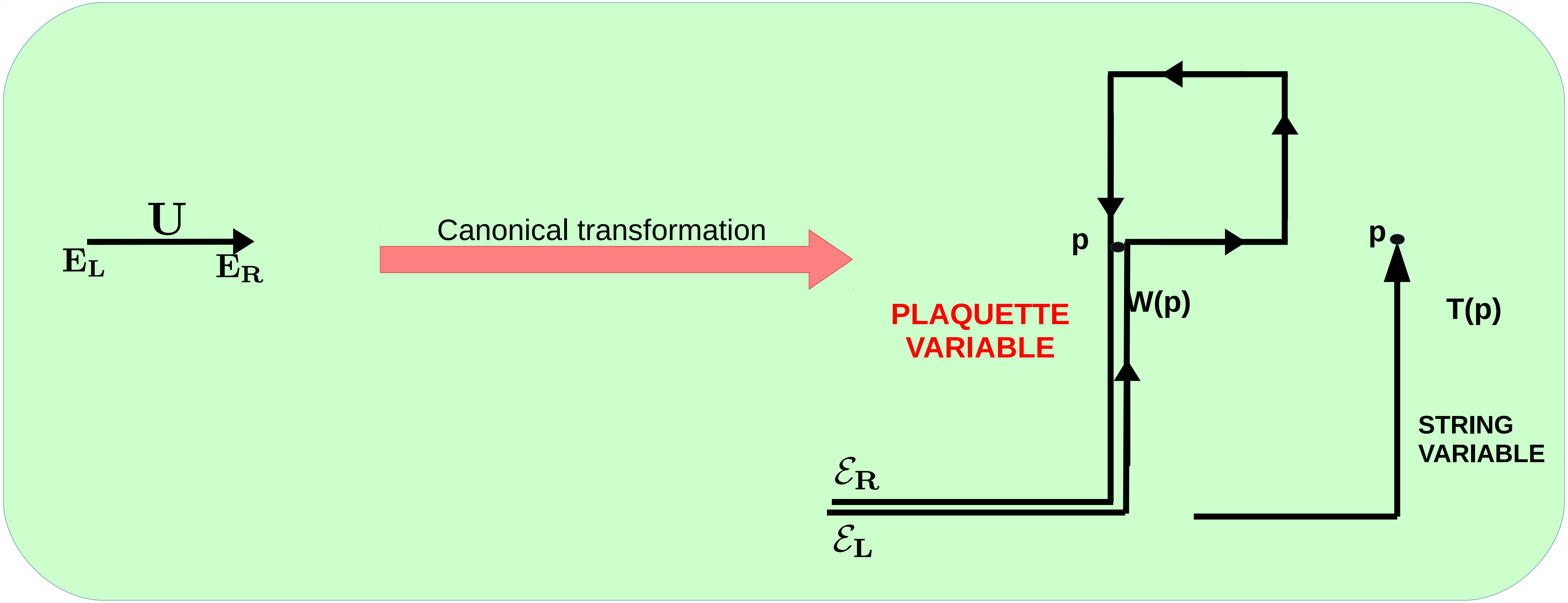
$$\mathbf{W}(\mathbf{2}) = \mathbf{U}(\mathbf{1})\mathbf{U}(\mathbf{2}) \qquad \mathbf{W}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{2}) = \mathbf{E}^{\mathbf{a}}_{\mathbf{R}}(\mathbf{2})$$

$$[E^\beta_L, U_{ij}] = -\tau^\beta_{ik} U_{kj}$$
$$[E^\beta_R, U_{ij}] = U_{ik}\tau^\beta_{kj}$$
$$[E^\alpha_L, E^\beta_L] = i f^{\alpha\beta\gamma} E^\gamma_L$$

Commutation relations preserved!

$$[W^\beta_L, W_{ij}] = -\tau^\beta_{ik} W_{kj}$$
$$[W^\beta_R, W_{ij}] = W_{ik}\tau^\beta_{kj}$$
$$[W^\alpha_L, W^\beta_L] = i f^{\alpha\beta\gamma} W^\gamma_L$$

Plaquette and string variables.



LOCAL GAUSS LAW → **GLOBAL GAUSS LAW** $\sum_{\mathbf{p}} \mathcal{E}_{\mathbf{L}}(\mathbf{p}) + \mathcal{E}_{\mathbf{R}}(\mathbf{p}) = \mathbf{0}$

- Plaquette variables form a complete set of operators.
- No mandelstam constraints!

Hydrogen atom symmetry revisited..

Hydrogen atom Symmetry group is given by **SO(4) ~ SU(2) x SU(2)** (bound states)

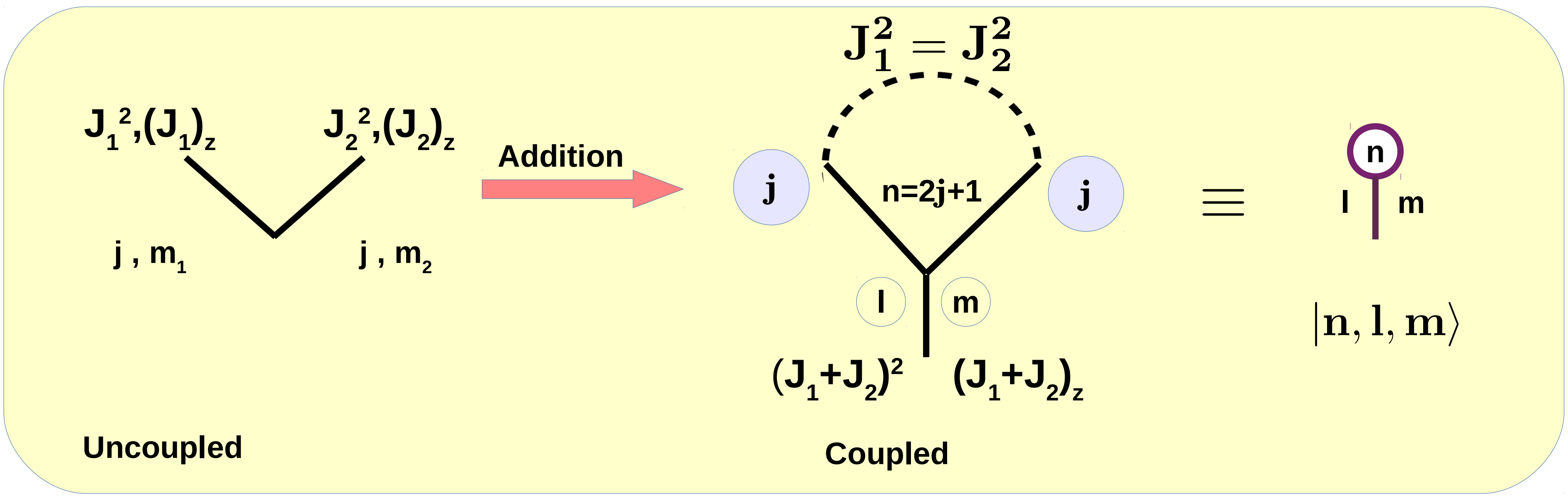
Hydrogen atom su(2) x su(2) algebra

$$\vec{\mathbf{J}}_1 = \frac{1}{2}\left(\vec{\mathbf{L}} + \vec{\mathbf{A}}\right) \qquad \vec{\mathbf{J}}_2 = \frac{1}{2}\left(\vec{\mathbf{L}} - \vec{\mathbf{A}}\right)$$

Where $\vec{\mathbf{L}}$ - Angular momentum

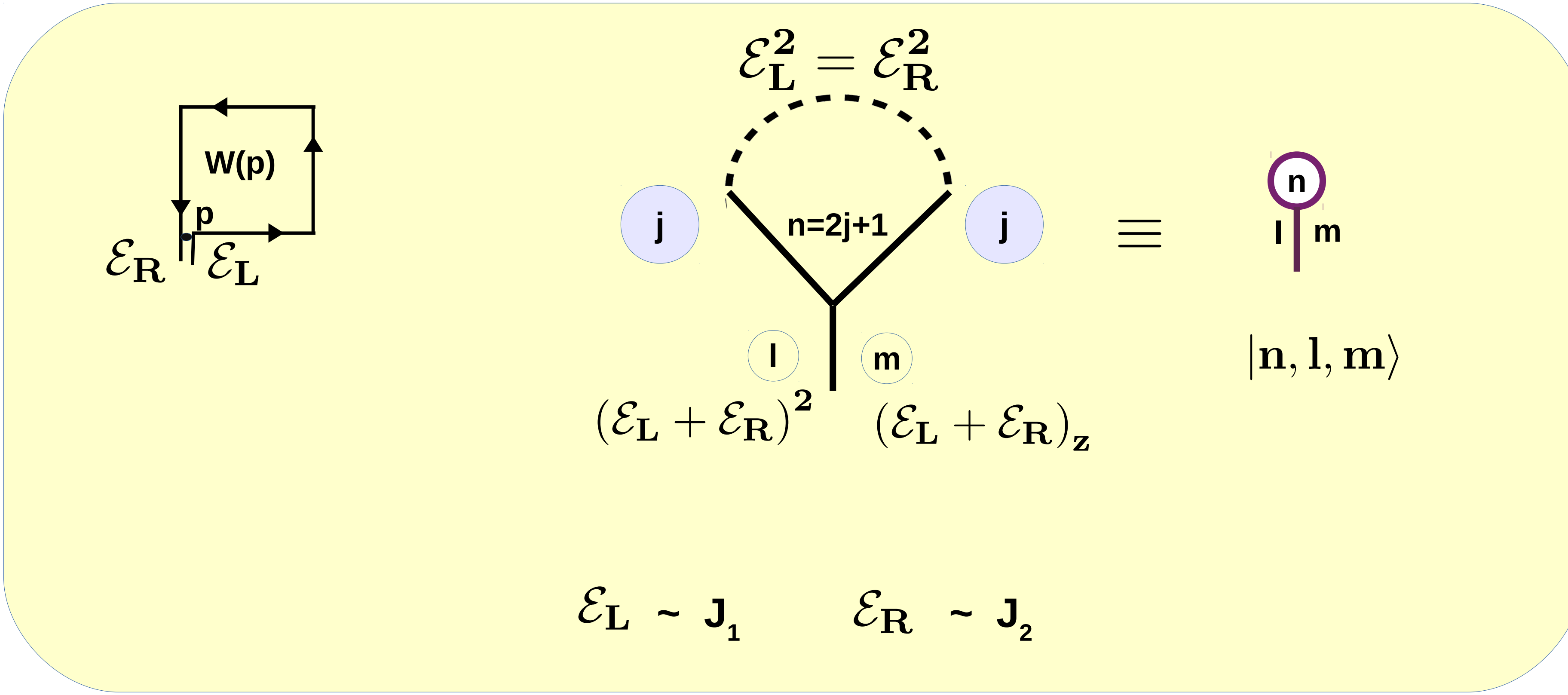
$\vec{\mathbf{A}}$ - Runge-Lenz vector

$$\vec{\mathbf{L}}\cdot\vec{\mathbf{A}} = 0 \implies \mathbf{J}_1^2 = \mathbf{J}_2^2$$



$|\mathbf{n}, \mathbf{l}, \mathbf{m}\rangle$ are eigenstates of H with energy $\mathbf{E_n} = \dfrac{-1}{2\mathbf{n}^2}$

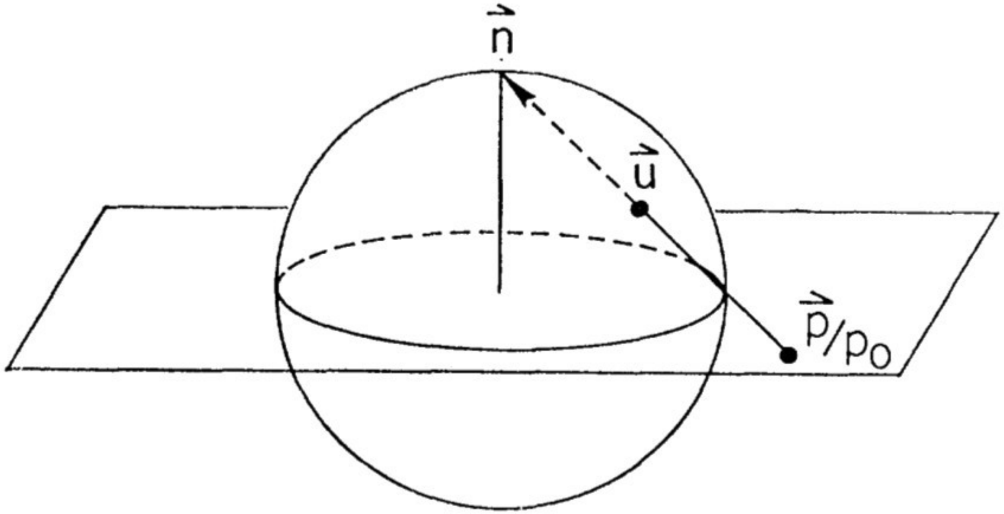
SU(2) LGT and H atoms*



$|\mathbf{n}, \mathbf{l}, \mathbf{m}\rangle$ identified with Hydrogen atom bound states.

* From lattice gauge theories to Hydrogen atoms – arxiv:1410.3318[hep-lat]

Hydrogen and S^3

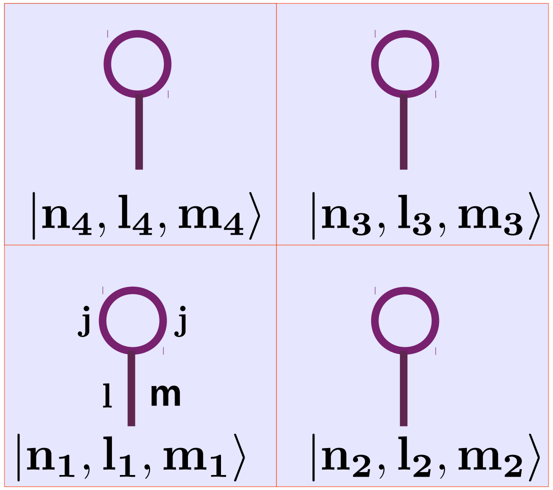


- SO(4) symmetry can be made manifest by transcribing the dynamics on S^3 onto which momentum coordinates are stereographically projected.
- Energy wavefunctions are hyperspherical harmonics on this S^3.
- This S^3 is identified with the group manifold of SU(2) holonomy.

* picture taken from : Bander , Itzykson – Reviews of modern physics, 38,330(1966)

Gauge invariant basis

Each plaquette is associated with a Hydrogen atom.

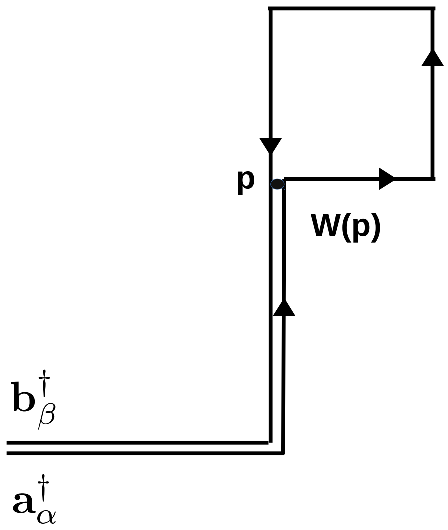


The only Gauss law on the entire lattice is :

$$\vec{\mathbf{l}}_1 + \vec{\mathbf{l}}_2 + \vec{\mathbf{l}}_3 + \vec{\mathbf{l}}_4 = 0$$

SU(2) Lattice gauge theory Hilbert space ~ coupled Hydrogen atoms with no net angular momenta.

Prepotential formulation and invariant algebra



$$\mathcal{E}^a_L(p) \equiv a^\dagger(p)\frac{\sigma^a}{2}a(p)$$

$$\mathcal{E}^a_R(p) \equiv b^\dagger(p)\frac{\sigma^a}{2}b(p)$$

$$W_{\alpha\beta} = \frac{1}{\sqrt{k_0}}(a^\dagger_\alpha \epsilon_{\beta\gamma} b^\dagger_\gamma + a_\alpha \epsilon_{\beta\gamma} b_\gamma)\frac{1}{\sqrt{k_0}}$$

$$k_0 = \frac{1}{2}\left(a^\dagger \cdot a + b^\dagger \cdot b\right) + 1$$

$$\mathrm{Tr}W = \frac{1}{\sqrt{k_0}}\left(k_+ + k_-\right)\frac{1}{\sqrt{k_0}}$$

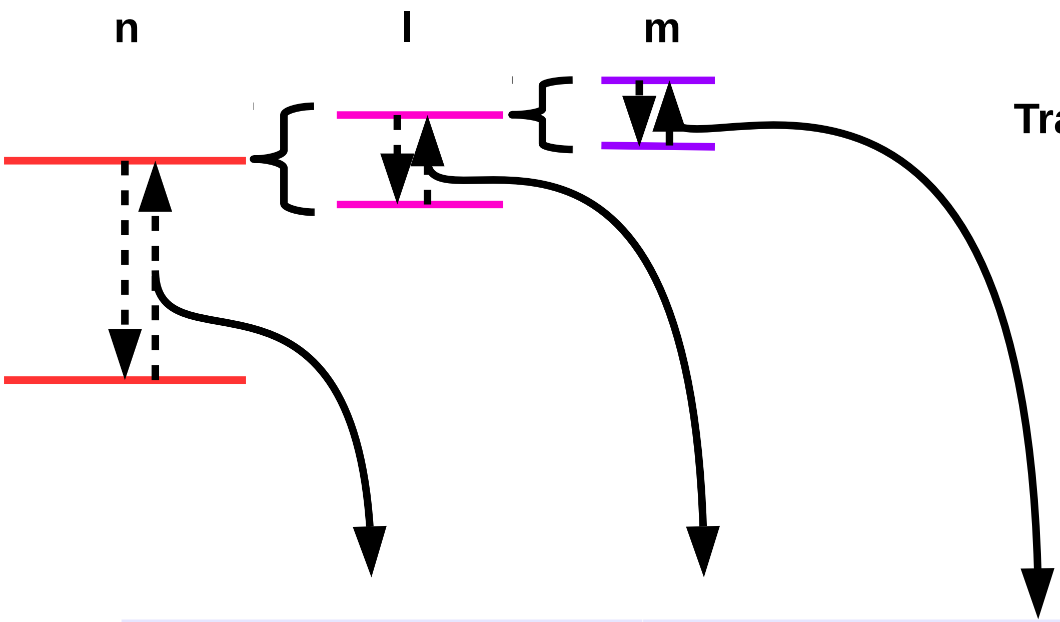
k^+, k^-, k_0 **Forms SU(1,1) algebra.**

k^0 **Counts the flux n**

k^+ **Increases n**

k^- **Decreases n**

SO(4,2) and SU(2) LGT.



Transition operators from $|n, l, m\rangle$ to $|n', l', m'\rangle$ is given by SO(4,2) generators.

	SO(4)	
$L_{46} = \frac{1}{2}(k^+ + k^-)$ $L_{45} = \frac{1}{2}(k^+ - k^-)$ $L_{56} = k_0$	$L_{ab} = \epsilon_{abc}(\mathcal{E}^c_R + \mathcal{E}^c_L)$ $L_{a4} = (\mathcal{E}^a_R - \mathcal{E}^a_L)$	$L_{a5} = \frac{1}{2}(a^\dagger\sigma_i\epsilon b^\dagger + a\epsilon\sigma_i b)$ $L_{a6} = \frac{1}{2}(a^\dagger\sigma_i\epsilon b^\dagger - a\epsilon\sigma_i b)$
Magnetic field operators: Counts and changes n	Electric field operators: Changes l,m	Fuses 2 loops.

SO(4,2) GENERATORS.

The Hamiltonian

$$\mathbf{H} = \sum_{<\mathbf{pp'}>} \mathbf{g^2}\{\mathcal{E}(\mathbf{p}) + \mathcal{E}(\mathbf{p'})\}^{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{g^2}}\mathbf{TrW}(\mathbf{p}) + \mathbf{O}(\mathbf{g^3})$$

Non local terms

$$\sum_{<\mathbf{pp'}>} \mathbf{g^2}\left\{4\mathcal{E}^{\mathbf{2}}(\mathbf{p}) + \mathcal{E}(\mathbf{p}) \cdot \mathcal{E}(\mathbf{p'})\right\}$$

Where < pp'> denotes the nearest neighbouring plaquettes

In terms of SO(4,2) generators:

$$\mathbf{H} = \mathbf{g^2}\left[\frac{\mathbf{1}}{\mathbf{2}}\epsilon_{\mathbf{abc}}\{\mathbf{L_{ab}}(\mathbf{p}) + \mathbf{L_{ab}}(\mathbf{p'})) - (\mathbf{L_{a4}}(\mathbf{p}) - \mathbf{L_{a4}}(\mathbf{p'})\}\right]^{\mathbf{2}} +$$

$$\frac{\mathbf{1}}{\mathbf{g^2}}\left[\mathbf{2} - \frac{\mathbf{1}}{\sqrt{\mathbf{L_{56}}}}\mathbf{L_{46}}\frac{\mathbf{1}}{\sqrt{\mathbf{L_{56}}}}\right] + \mathbf{O}(\mathbf{g^3})$$

Ground state.

Variational method:

Variational ansatz for ground state : collection of uncoupled H atoms with l=m=0.

$$|\psi_0\rangle = e^{S}|0\rangle \qquad\qquad S = \sum_{p} \lambda \left(k^{+}(p) + k^{-}(p) \right)$$

Minimizing $\langle\psi_0|H|\psi_0\rangle \implies \lambda(g) \sim \left(\frac{1}{g^2}\right) \quad \text{for} \quad g^2 \to 0$

$$|\psi_1\rangle = \sum_{p} k^{+}(p)|\psi_0\rangle - \langle\psi_0|k_{+}(p)|\psi_0\rangle$$

Mass gap $\langle\psi_1|H|\psi_1\rangle - \langle\psi_0|H|\psi_0\rangle$ **~ g^2 expected.**

Conclusion.

- Correspondence between Collection of wigner coupled Hydrogen atoms and Pure SU(2) lattice gauge theory Hilbert space is established.
- The issue of mandelstam constraints in loop formulation is completely bypassed.
- Hamiltonian is found to be that of nearest neighbouring spin interactions in weak coupling limit.
- Dynamics is governed by SO(4,2) generators.

THANK YOU .

The exact Non-local Hamiltonian

$$
\begin{aligned}
H &= \sum_{m,n\in\Lambda} \Bigg\{ g^2 \Big[\vec{\mathcal{E}}_L(m,n) + \vec{\mathcal{E}}_R(m,n-1) + \Delta_{XY}(m,n)\Big]^2 \\
&+ g^2 \Big[\vec{\mathcal{E}}_R(m,n) + R\,(\mathcal{W})\,\vec{\mathcal{E}}_L(m-1,n) + \Delta_Y(m,n)\Big]^2 \\
&+ \frac{1}{g^2}\Big(2 - Tr\ \mathcal{W}(m,n)\Big)\Bigg\}. \qquad (1)
\end{aligned}
$$

$$
\Delta^a_{XY}(m,n) \equiv \delta_{n,0} \sum_{r=m+1}^{N} \sum_{s=0}^{N} E^a_L(r,s) + E^a_R(r,s),
$$

$$
\Delta^a_Y(m,n) \equiv \sum_{s=(n+1)}^{N} E^a_L(m,s) + E^a_R(m,s), \;\; R(\mathcal{W}) \equiv \prod_{q=0}^{n-1} R\,(\mathcal{W}(m,q))
$$

The exact Non-local Hamiltonian

$$H = \sum_{m,n\in\Lambda} \Big\{ g^2 \Big[\vec{\mathcal{E}}_L(m,n) + \vec{\mathcal{E}}_R(m,n-1) + \Delta_{XY}(m,n)\Big]^2 + g^2 \Big[\vec{\mathcal{E}}_R(m,n) + R\,(\mathcal{W})\,\vec{\mathcal{E}}_L(m-1,n) + \Delta_Y(m,n)\Big]^2 + \frac{1}{g^2}\Big(2 - Tr\ \mathcal{W}(m,n)\Big) \Big\}. \qquad (1)$$

$$\Delta^a_{XY}(m,n) \equiv \delta_{n,0} \sum_{r=m+1}^{N} \sum_{s=0}^{N} E^a_L(r,s) + E^a_R(r,s),$$

$$\Delta^a_Y(m,n) \equiv \sum_{s=(n+1)}^{N} E^a_L(m,s) + E^a_R(m,s), \;\; R(\mathcal{W}) \equiv \prod_{q=0}^{n-1} R\,(\mathcal{W}(m,q))$$