SAT+SMT School, TIFR, 4–10 December 2016

Tutorial on Propositional Logic, 4 December 2016

Exercises

1. Prove the claim that for an assignment $\mathcal{A}$ and a formula F, $\mathcal{A}(F)$ depends only on the vocabular of F—the atomic propositions mentioned in F.

2. The binary connective *nand*, $F \downarrow G$ is defined by the truth table corresponding to $\neg(F \wedge G)$. Show that *nand* is complete — that it can express all binary boolean connectives.

3. The binary connective *xor*, $F \oplus G$ is defined by the truth table corresponding to $(\neg F \wedge G) \vee (F \wedge \neg G)$. Show that *xor* is *not* complete — that it cannot express all binary boolean connectives.

4. Prove the Substitution Theorem:

   Let F be a subformula of H, and let $F \equiv G$. Let H' be the formula obtained by replacing F in H with G. Then $H \equiv H'$.

5. Complete the proof that Tseitsin encoding to CNF of a formula F is equisatisfiable with F.

6. Every formula in CNF can be converted to an equivalent formula in 3-CNF, where each clause has at most 3 literals. This can be done by introducing new propositions to split large clauses. Formalize this idea. What is the complexity of the resulting formula?

7. The Pigeonhole Principle states that if we try to place $n+1$ pigeons in n pigeonholes, at least one pigeonhole must house more than one pigeon. Express this in propositional logic. What is the complexity of your formula as a function of n?

8. Complete the proof by induction, of the following statement.

   $$\text{If } F \text{ is unsatisfiable, then } \emptyset \in Res^*(F).$$

9. Consider the following game, due to Raymond Smullyan. The game is played with balls labelled with positive integers. You have an unlimited supply of balls. There is no restriction on how many balls can have the same number or how big the numbers can be.

   The game proceeds in stages. At stage 0, you are given a box containing one ball. The game goes from stage n to stage $n+1$ by playing one of the following moves:

   (a) Take one ball out of the box without putting any ball back into the box.

   (b) Take one ball out of the box labelled, say, m. Put back an arbitrary, but finite, number of balls labelled with positive integers strictly smaller than m.

   For instance, if you take out a ball labelled 36, you may replace it by no ball, or by one billion balls labelled 35, or one ball labelled 10, or (35 balls labelled 35 + 34 balls labelled 34 + $\cdots$ + 1 ball labelled 1) or ...

   Use König's Lemma to prove that, no matter how you play at each stage of the game, you will eventually be forced to empty the box.

   (*Hint*: Focus on a single instance of the game. Consider any sequence of moves and construct a suitable tree in which the vertices are all the balls that are used in those moves.)