First Order Logic: A Brief Tour (Part I)

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Notation

- Variables: $x, y, z, \ldots$
  - Represent elements of an underlying set
- Constants: $a, b, c, \ldots$
  - Specific elements of underlying set
- Function symbols: $f, g, h, \ldots$
  - *Arity* of function: # of arguments
  - 0-ary functions: constants
- Relation (predicate) symbols: $P, Q, R, \ldots$
  - Hence, also called "predicate calculus"
  - *Arity* of predicate: # of arguments
- Fixed symbols:
  - Carried over from prop. logic: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, (,)
  - New in FOL: $\exists$, $\forall$ ("quantifiers")

Equality in FOL

- A special binary predicate, used widely in maths
- Represented by special predicate symbol "="
- Semantically, binary identity relation (more on this later ...)
- First-order logic with equality
  - Different expressive power vis-a-vis first-order logic
  - Most of our discussions will assume availability of "="
  - Refer to as "first-order logic" unless the distinction is important

Syntax of FOL

Two classes of syntactic objects: *terms* and *formulas*

Terms

- Every variable is a term
- If f is an m-ary function, $t_1, \ldots t_m$ are terms, then $f(t_1, \ldots t_m)$ is also a term

Atomic formulas

- If R is an n-ary predicate, $t_1, \ldots t_n$ are terms, then $R(t_1, \ldots t_m)$ is an atomic formula
- Special case: $t_1 = t_2$

Syntax of FOL

- *Primitive* fixed symbols: $\wedge$, $\neg$, $\exists$
  - Other choices also possible: E.g., $\vee$, $\neg$, $\forall$

Rules for formuling formulas

- Every atomic formula is a formula
- If φ is a formula, so are $\neg\varphi$ and (φ)
- If φ_1 and φ_2 are formulas, so is $\varphi_1 \wedge \varphi_2$
- If φ is a formula, so is $\exists x\, \varphi$ for any variable x

- Formulas with other fixed symbols definable in terms of formulas with primitive symbols.
  - $\varphi_1 \vee \varphi_2 \triangleq \neg(\neg\varphi_1 \wedge \neg\varphi_2)$
  - $\varphi_1 \rightarrow \varphi_2 \triangleq \neg\varphi_1 \vee \varphi_2$
  - $\varphi_1 \leftrightarrow \varphi_2 \triangleq (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$
  - $\forall x\, \varphi \triangleq \neg(\exists x\, \neg\varphi)$

FOL formulas as strings

- Alphabet (over which strings are constructed):
  - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
  - Set of constants, functions, predicates, e.g. $\{a, b, f, =, P\}$
  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Well-formed formula: string formed according to rules on prev. slide
  - $\forall x_1 (\forall x_2 (((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$ is well-formed
  - $\forall(\forall x_1 (x_1 = ab)\neg()x_2)$ is not well-formed
- Well-formed formulas can be represented using parse trees
  - Consider the rules on prev. slide as production rules in a context-free grammar

Vocabulary

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  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Smallest vocabulary to generate $\forall x_1 (\forall x_2 (((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$?

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  - $\{a, b, f, =\}$

Free Variables in a Formula

Free variables are those that are not quantified in a formula.
Let free(φ) denote the set of free variables in φ

- If φ is an atomic formula, free(φ) $= \{x \mid \text{x occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, free(φ) $=$ free(ψ)
- If $\varphi = \varphi_1 \wedge \varphi_2$, free($\varphi$) $=$ free(φ_1) $\cup$ free(φ_2)
- if $\varphi = \exists x\, \varphi_1$, free($\varphi$) $=$ free(φ_1) $\setminus \{x\}$

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  - $=$ free($(\exists x\, P(x, y))$) $\cup$ free($\forall y\, Q(x, y)$)
  - $=$ free($P(x, y)$) $\setminus \{x\}$ $\cup$ free($Q(x, y)$) $\setminus \{y\}$
  - $= \{x, y\} \setminus \{x\} \cup \{x, y\} \setminus \{y\} = \{x, y\}$

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  - $= \{x,y\} \setminus \{x\} \;\cup\; \{x,y\} \setminus \{y\} \;=\; \{x,y\}$

If φ has free variables $\{x,y\}$, we write $\varphi(x,y)$
A formula with no free variables is a **sentence**, e.g. $\exists x \forall y\, f(x) = y$

Bound Variables in a Formula

Bound variables are those that are quantified in a formula.
Let $\text{bnd}(\varphi)$ denote the set of bound variables in φ

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- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{bnd}(\varphi) = \text{bnd}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{bnd}(\varphi) = \text{bnd}(\varphi_1) \cup \text{bnd}(\varphi_2)$
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  - $= \text{bnd}((\exists x\, P(x, y))) \;\cup\; \text{bnd}(\forall y\, Q(x, y))$
  - $= \text{bnd}(P(x, y)) \cup \{x\} \;\cup\; \text{bnd}(Q(x, y)) \cup \{y\}$
  - $= \emptyset \cup \{x\} \;\cup\; \emptyset \cup \{y\}$
  - $= \{x\} \;\cup\; \{y\} = \{x, y\}$!!!
- $\text{free}(\varphi)$ and $\text{bnd}(\varphi)$ are not complements!

Substitution in FOL

Suppose $x \in \text{free}(\varphi)$ and t is any term.
We wish to replace every free occurrence of x in φ with t, such that free variables in t stay free in the resulting formula.

Term t is free for x in φ if no free occurrence of x in φ is in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

- $\varphi \triangleq \exists y\, R(x, y) \vee \forall x\, R(z, x)$, and t is $f(z, x)$
- $f(z, x)$ is free for x in φ, but $f(y, x)$ is not

$\varphi[t/x]$: Formula obtained by replacing each free occurrence of x in φ by t, if t is free for x in φ

- For φ defined above,
  $\varphi[f(z, x)/x] \triangleq \exists y\, R(f(z, x), y) \vee \forall x\, R(z, x)$

Semantics of FOL: Some Intuition

$$\varphi \triangleq \forall x \forall y \, (P(x,y) \rightarrow \exists z \, (\neg(z = x) \wedge \neg(z = y) \wedge P(x,z) \wedge P(z,y))$$

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English reading: For every x and y, if $P(x,y)$ holds, we can find z distinct from x and y such that both $P(x,z)$ and $P(z,y)$ hold.

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Case 1:

- Variables take values from **real numbers**
- $P(x,y)$ represents $\mathbf{x} < \mathbf{y}$
- English reading simply states "real numbers are dense"
- φ is **true**

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Case 2:

- Variables take values from **real numbers**
- $P(x, y)$ represents $\mathbf{x} \leq \mathbf{y}$
- English reading requires the following to be true
  - If $x = y$, there is a z such that $z \neq x$, $x \leq z$ and $z \leq x$
  - Thus, $z \neq x$ and $z = x$
- φ is **false**

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Case 3:

- Variables take values from **natural numbers**
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- English reading states that "natural numbers are dense"
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- Variables take values from **natural numbers**
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Truth of φ depends on the underlying set from which variables take values, and on how constants, functions, predicates are interpreted

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Vocabulary $\mathcal{V}$:   E.g. $\mathcal{V} : \{a, f, =, R\}$

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     e.g. Interp. for $=$: $\{(c, c) \mid c \in \mathbb{N}\}$ – fixed interpretation
     Interp. for $R : \{(c, d) \mid c, d \in U, c < d\}$

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1 and 2 define a $\mathcal{V}$-**structure** $M = (U^M, (a^M, f^M, R^M))$

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3. *Binding (aka environment)* $\alpha : \text{free}(\varphi) \rightarrow U$
   e.g. $\alpha(y) = 2$

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Given structure M and binding α, does φ evaluate to **true**?
Notationally, does $\mathbf{M}, \alpha \models \varphi$?

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- Extend $\alpha : \mathsf{free}(\varphi) \rightarrow U^M$ to $\overline{\alpha} : \mathit{Terms}(\varphi) \rightarrow U^M$
  - If t is a variable x, $\overline{\alpha}(t) = \alpha(x)$
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    - In prev. example, suppose $\alpha'(x) = 1, \alpha'(y) = 2$. Then $M, \alpha' \models R(x, f(y, a))$ as $(\overline{\alpha'}(x), \overline{\alpha'}(f(y, a))) = (1, 2) \in R^M$.

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- $M, \alpha \models \neg\varphi_i$ iff $M, \alpha \not\models \varphi_1$

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- $M, \alpha \models \neg\varphi_i$ iff $M, \alpha \not\models \varphi_1$
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Semantics of FOL: Formalizing the intuition

Given structure M and binding α, does φ evaluate to **true**?
Notationally, does $\mathbf{M}, \alpha \models \varphi$?

- Extend $\alpha : \text{free}(\varphi) \rightarrow U^M$ to $\overline{\alpha} : Terms(\varphi) \rightarrow U^M$
  - If t is a variable x, $\overline{\alpha}(t) = \alpha(x)$
  - If t is $f(t_1, \ldots t_m)$, $\overline{\alpha}(t) = f^M(\overline{\alpha}(t_1), \ldots \overline{\alpha}(t_m))$
    - In prev. example, $\overline{\alpha}(f(y, a)) = f^M(\alpha(y), a^M) = 2 + 0 = 2$
- If φ is an atomic formula
  - $M, \alpha \models (t_1 = t_2)$ iff $\overline{\alpha}(t_1)$ and $\overline{\alpha}(t_2)$ coincide
  - $M, \alpha \models P(t_1, \ldots t_m)$ iff $(\overline{\alpha}(t_1), \ldots \overline{\alpha}(t_m)) \in P^M$
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- $M, \alpha \models \exists x\, \varphi$ iff there is some $c \in U^M$ such that $M, \alpha[x \mapsto c] \models \varphi$, where
  - $\alpha[x \mapsto c](v) = \alpha(v)$, if variable v is different from x
  - $\alpha[x \mapsto c](x) = c$

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- Finally, $M, \alpha \models \varphi$
- Note that if $\alpha'(y) = 1$, $M, \alpha' \not\models \varphi$

Semantic Relations in FOL

Let $\Gamma = \{\varphi_1, \varphi_2, \ldots\}$ be a (possibly infinite) set of formulas, and ψ be a formula

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- **Semantic Entailment:** $\Gamma \models \psi$ holds iff whenever $M, \alpha \models \varphi_i$ for all $\varphi_i \in \Gamma$, then $M, \alpha \models \psi$ as well.
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  - $\forall x\, P(x, y) \rightarrow \exists x\, P(x, y)$ is valid
- **Consistency:** Γ is consistent iff there is at least one M and α such that $M, \alpha \models \varphi_i$ for all $\varphi_i \in \Gamma$.
  - $\{\exists x\, R(x, y),\ \exists x\, R(f(x), y),\ \exists x\, R(f(f(x)), y), \ldots\}$ is consistent

Sematic Equivalence in FOL

$\varphi \equiv \psi$ iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Quantifier Equivalences

- $\forall x \forall y\, \varphi \;\equiv\; \forall y \forall x\, \varphi, \quad \exists x \exists y\, \varphi \;\equiv\; \exists y \exists x\, \varphi$
- $\forall x\, (\varphi_1 \wedge \varphi_2) \;\equiv\; (\forall x\, \varphi_1) \wedge (\forall x\, \varphi_2)$
- $\exists x\, (\varphi_1 \vee \varphi_2) \;\equiv\; (\exists x\, \varphi_1) \vee (\exists x\, \varphi_2)$
- If $x \notin \mathsf{free}(\varphi_2)$, then $Qx\, (\varphi_1 \text{ op } \varphi_2) \;\equiv\; (Qx\, \varphi_1) \text{ op } \varphi_2$, where $Q \in \{\exists, \forall\}$ and op $\in \{\vee, \wedge\}$.

Renaming Quantified Variables

Let $z \notin \mathsf{free}(\varphi) \cup \mathsf{bnd}(\varphi)$.
Then $Qx\, \varphi \;\equiv\; Qz\, \varphi[z/x]$ for $Q \in \{\exists, \forall\}$.

Enabler for substitution, e.g., $\exists x\, R(f(x,y), w) \;\equiv\; \exists z\, R(f(z,y), w)$
$f(x,y)$ not free for y in $\exists x\, R(f(x,y), w)$, but is free for y in $\exists z\, R(f(z,y), w)$.

First-order Definable Structures

- If φ is a $\mathcal{V}$-sentence (no free vars), no binding α necessary for evaluating truth of φ
  - Given $\mathcal{V}$-structure M, we can ask if $M \models \varphi$
  - Class of $\mathcal{V}$-structures defined by φ is $\{M \models \varphi\}$
- Some examples of structures: graphs, databases, number systems

Graphs as FO structures

A graph G

- U^G: set of vertices
- Vocabulary $\mathcal{V}$: $\{E, =\}$, where E is a binary (edge) relation
- Interpretation: For $a, b \in U^G$, $E^G(a, b) = \textbf{true}$ iff there is an edge from vertex a to vertex b in G

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- $\forall x \forall y\, (\neg(x = y) \rightarrow E(x, y))$
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- $\forall x \forall y \forall z \, (\neg(x = y) \wedge \neg(y = z) \wedge \neg(z = x)) \rightarrow \neg(E(x, y) \wedge E(y, z) \wedge E(z, x))$
  - (Infinite) class of all graphs with no cycles of length 3

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- $\exists x \exists y\, (\neg(x = y) \wedge E(x, y) \wedge \forall z\, ((x = z) \vee (y = z)))$
  - (Finite) class of graphs with exactly two connected vertices.

Relational Databases as FO structures

A relational database D

- U^D: set of (possibly differently typed) data items
- Vocabulary $\mathcal{V}$: $\{P_1, \ldots P_k, =\}$, where P_i is a k_i-ary predicate corr. to the i^{th} table in database with k_i columns
- Interpretation: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) =$ **true** iff $(a_1, \ldots a_{k_1})$ is a row of the i^{th} table

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Example database query:

- $$\varphi(x) \triangleq \exists y \exists z\, (\mathrm{Dob}(x, y) \wedge \mathrm{After}(y, \text{“}01/01/1990\text{''}) \wedge \mathrm{Class}(x, z) \wedge \mathrm{Primary}(z))$$

  Defines set of students born after “01/01/1990” and studying in a primary class.

Number systems as FO structures

Natural/real numbers with addition, multiplication, linear ordering and constants $\mathbf{0}$ and $\mathbf{1}$ (fixed interpretation)

- $\mathfrak{N} = (\mathbb{N}, \mathbf{0}, \mathbf{1}, \times, +, <, =)$
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Examples of properties expressible in FOL:

- $\mathfrak{R} \models \forall x \exists y\, (x = ((y \times y) \times y))$
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  - Not every natural number can be expressed as the sum of squares of two natural numbers. This can be done for real numbers

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- $\mathfrak{N} \models \forall x \exists y \, ((x < y) \wedge (\forall z \forall w \, (y = z \times w) \rightarrow ((z = y) \vee (w = y)))$
  - There are infinitely many prime natural numbers

A Proof System for FOL

All proof rules considered yesterday for Propositional Logic

Additional proof rules for quantifiers and equality

$$\frac{}{t = t} \quad \text{(= introduction)}$$

$$\frac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]} \quad \text{(= elimination)}$$

$$\frac{\forall x\, \varphi}{\varphi[t/x]} \quad (\forall \text{ elimination})$$

$$\frac{[x_0 \quad \cdots \quad \varphi[x_0/x]]}{\forall x\, \varphi} \quad (\forall \text{ introduction})$$

$$\frac{\varphi[t/x]}{\exists x\, \varphi} \quad (\exists \text{ introduction})$$

$$\frac{\exists x\, \varphi \qquad [x_0 \;\; \varphi[x_0/x] \quad \cdots \quad \chi]}{\chi} \quad (\exists \text{ elimination})$$

Soundness, Completeness, Undecidability

Let Γ be a finite set of FOL formulas, and ψ be a FOL formula. We say $\Gamma \vdash \psi$ if ψ can be syntactically derived from Γ by a finite sequence of application of our proof rules.

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If $\Gamma \vdash \psi$, then $\Gamma \models \psi$

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Undecidabilty

Given a FOL formula φ, checking validity of φ, i.e. does $\{\} \models \varphi$ is undecidable.

Completeness implies that detecting non-validity is non-terminating, in general

Compactness of FOL

Compactness Theorem

Let Γ be a (possibly infinite) set of FOL formulas. If all finite subsets of Γ are consistent (satisfiable), then so is Γ

Consequences of compactness:

- **Upward Lowenheim Skolem Theorem:** Let φ be a $\mathcal{V}$-sentence such that for every natural number $n \geq 1$, there is a $\mathcal{V}$-structure M_n with $\geq n$ elements in its universe, such that $M_n \models \varphi$. Then there exists a $\mathcal{V}$-structure M with inifnitely many elements in its universe, such that $M \models \varphi$
  Proof: Follows from Compactness Theorem
  Consequences:
  There is no FOL sentence that describes the class of of finite cliques
  There is no FOL sentence that describes the class of finite sets