SAT Tutorial

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The CDCL SAT disruption

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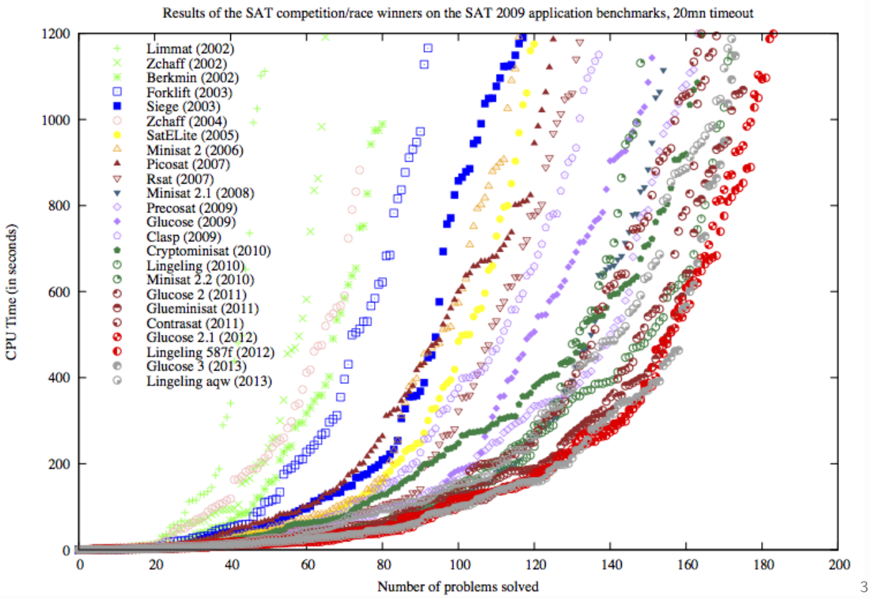
  - But, CDCL SAT solving is a success story of Computer Science
  - CDCL SAT solving has been truly disruptive
  - Hundreds (thousands?) of practical applications

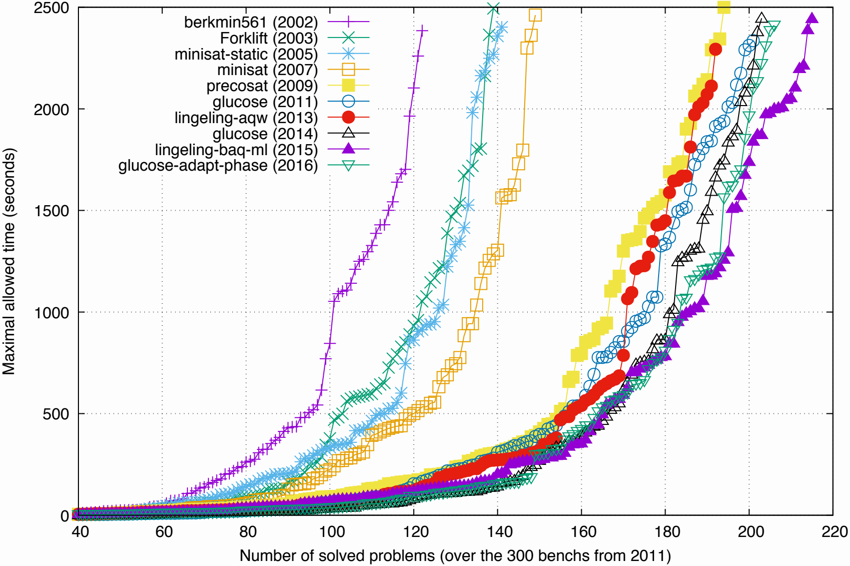
CDCL SAT solver improvement I

[Source: Le Berre 2013]

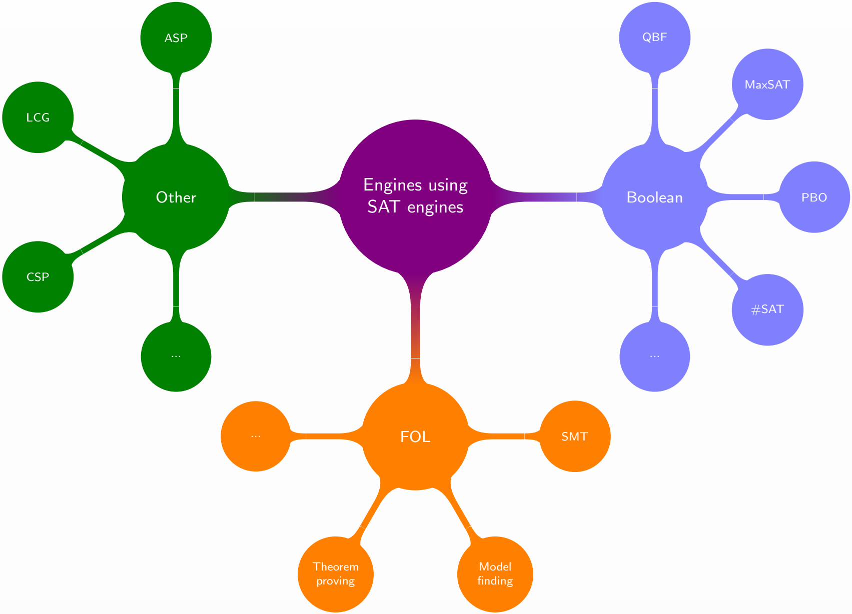


CDCL SAT solver improvement II

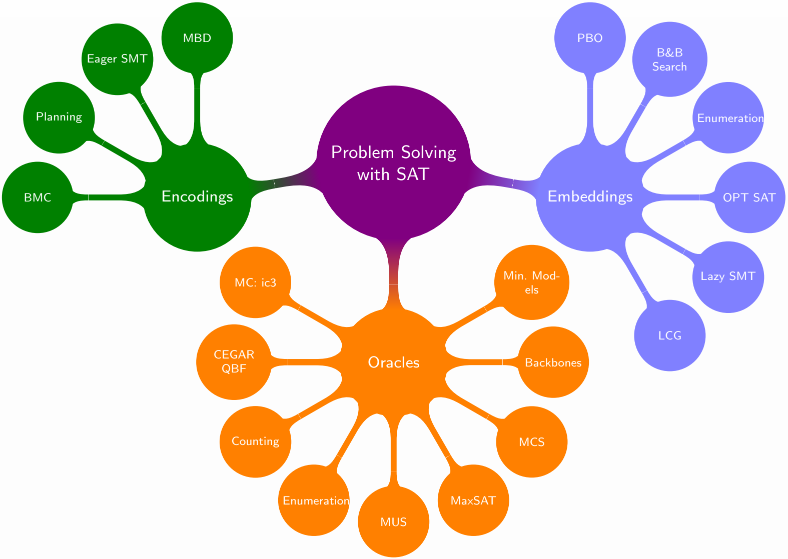
[Source: Simon 2015]



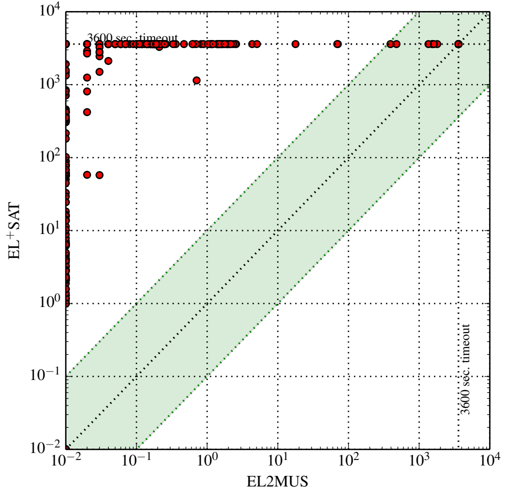
CDCL SAT is **the** engines' engine



CDCL SAT is ubiquitous in problem solving

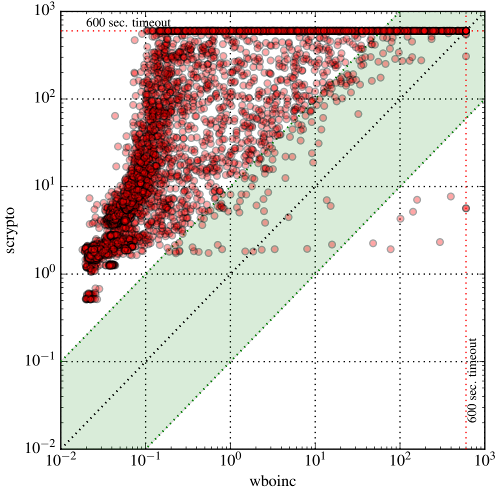


SAT can make the difference – axiom pinpointing



- $\mathcal{EL}^{+}$ medical ontologies [SAT'15]
  - Minimal unsatisfiability (MUSes) & maximal satisfiability (MCSes) & Enumeration

SAT can make the difference – model based diagnosis



- Model-based diagnosis problem instances [IJCAI'15]
  - Maximum satisfiability (MaxSAT)

This tutorial

- Part #1: Modern SAT solvers
  - **Conflict-Driven Clause Learning** (**CDCL**) SAT solvers
    - **Goal:** Overview for non-experts

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- Part #2: Modeling problems for SAT
  - Propositional encodings
  - Modeling examples

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- Part #1: Modern SAT solvers
  - **Conflict-Driven Clause Learning** (**CDCL**) SAT solvers
    - **Goal:** Overview for non-experts
- Part #2: Modeling problems for SAT
  - Propositional encodings
  - Modeling examples
- Part #3: Problem solving with SAT oracles
  - Minimal unsatisfiability (MUS)
  - Maximum satisfiability (MaxSAT)
  - ~~Maximal satisfiability~~ (MSS/MCS)
  - ~~Enumeration problems~~
  - ~~Counting problems~~
  - ~~Quantification problems~~
  - ~~Etc.~~

Part I

CDCL SAT Solving

Outline

Basic Definitions

Clause Learning, UIPs & Minimization

Search Restarts & Lazy Data Structures

Why CDCL Works?

Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
- Formula can be SAT/UNSAT

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- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
- Formula can be SAT/UNSAT
- Example:

$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

  - Example models:
    - $\{r, s, a, b, c, d\}$
    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

- Resolution rule: [DP60,R65]

$$\frac{(\alpha \vee x) \qquad\qquad (\beta \vee \bar{x})}{(\alpha \vee \beta)}$$

  - Complete proof system for propositional logic

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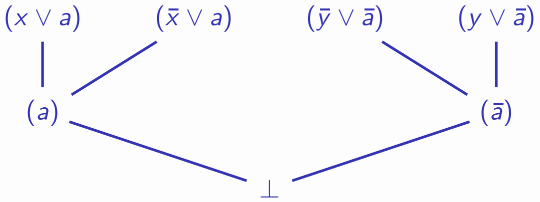

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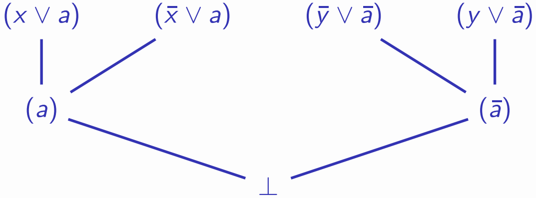

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- Self-subsuming resolution (with $\alpha' \subseteq \alpha$): [E.g. SP04,EB05]

$$\frac{(\alpha \vee x) \qquad\qquad (\alpha' \vee \bar{x})}{(\alpha)}$$

  - (α) subsumes $(\alpha \vee x)$

Unit propagation

$$\begin{aligned}\mathcal{F} &= (r) \wedge (\bar{r} \vee s) \wedge \\ &\quad (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \wedge \\ &\quad (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

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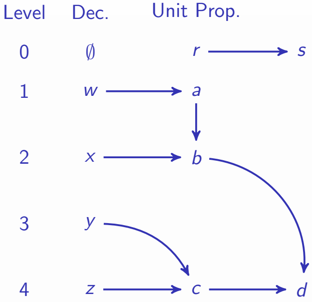

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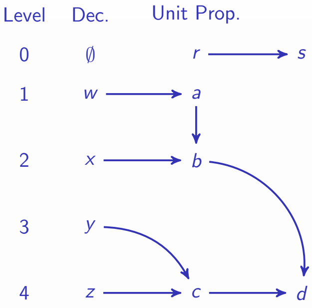
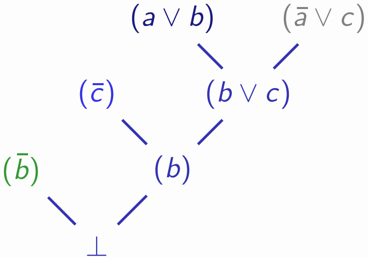


- Decisions / Variable Branchings: $w = 1, x = 1, y = 1, z = 1$

- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 @ 1$, $x = 1 @ 2$, $y = 1 @ 3$, $z = 1 @ 4$
    - $r = 1 @ 0$, $d = 1 @ 4$, ...

Resolution proofs

- Refutation of unsatisfiable formula by iterated resolution operations produces resolution proof
- An example:
  $\mathcal{F} = (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$
- Resolution proof:

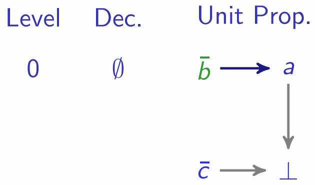


- A modern SAT solver can generate resolution proofs using clauses learned by the solver [ZM03]

Unsatisfiable cores & proof traces

- CNF formula:

$$\mathcal{F} \;=\; (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$$



Implication graph with conflict

Unsatisfiable cores & proof traces

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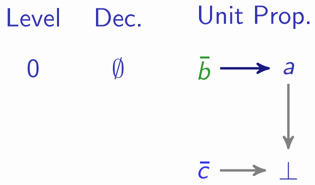

Proof trace $\bot$: $(\bar{a} \vee c)$ $(a \vee b)$ $(\bar{c})$ $(\bar{b})$

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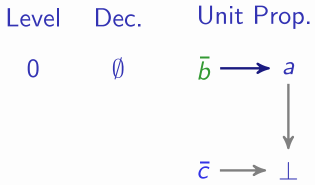

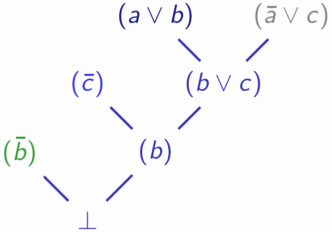

Resolution proof follows structure of conflicts

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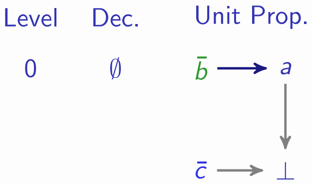
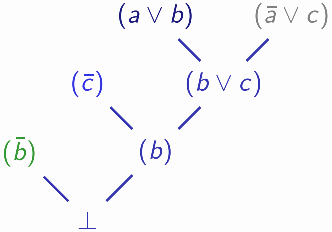
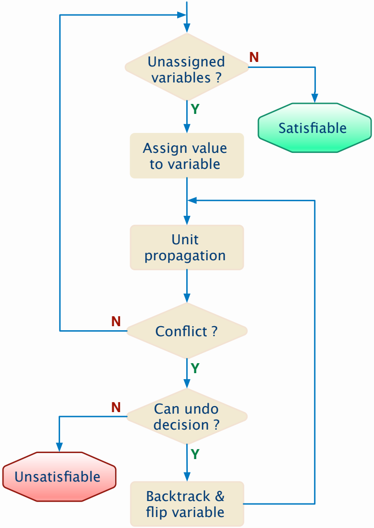




Unsatisfiable subformula (core): $(\bar{c}), (\bar{b}), (\bar{a} \vee c), (a \vee b)$

The DPLL algorithm

[DL60,DLL62]



- Optional: pure literal rule

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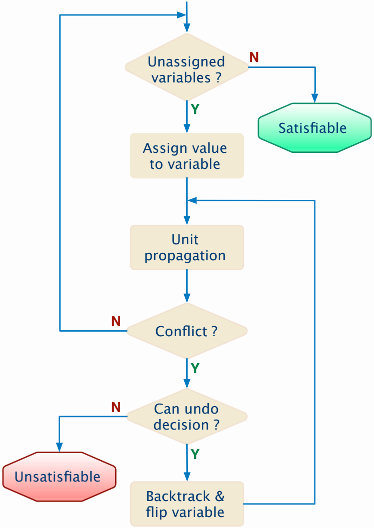

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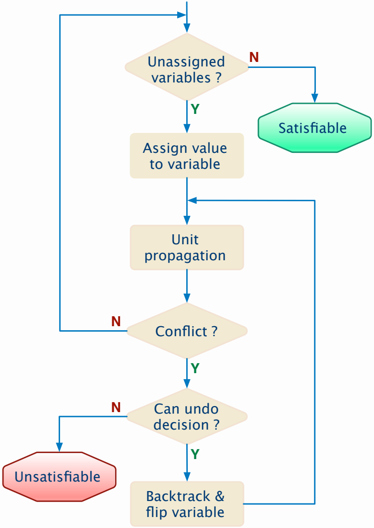

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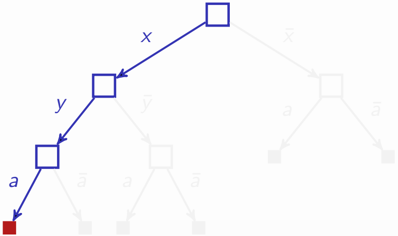
1	x	
2	y	
3	a	$\rightarrow b \rightarrow \bot$



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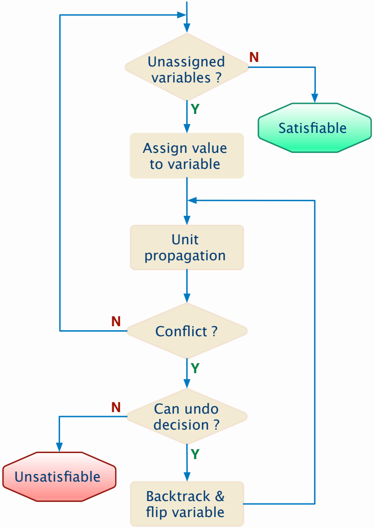

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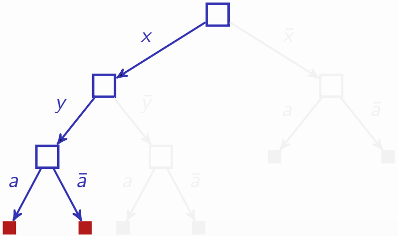

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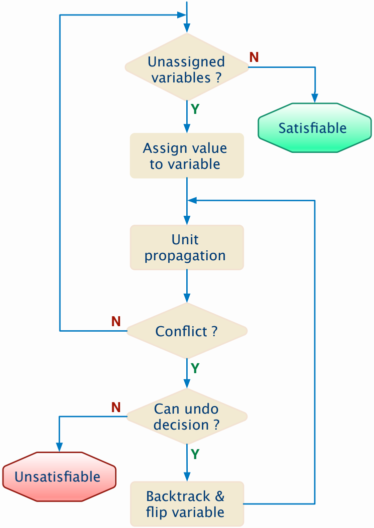

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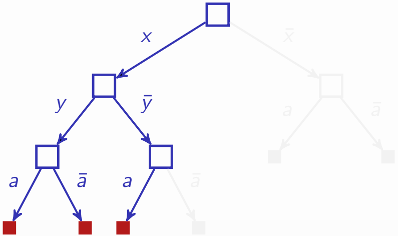

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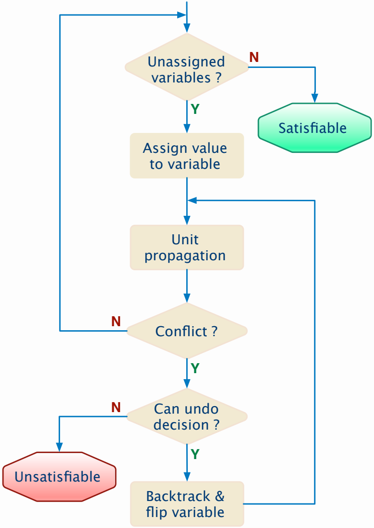

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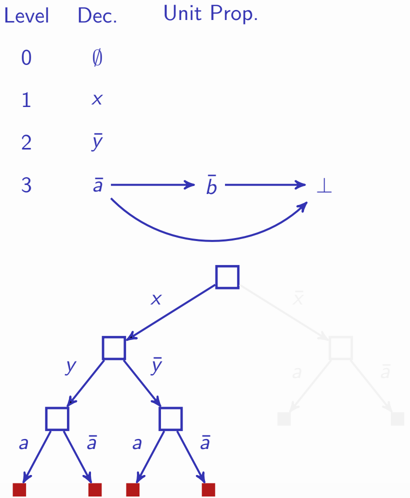

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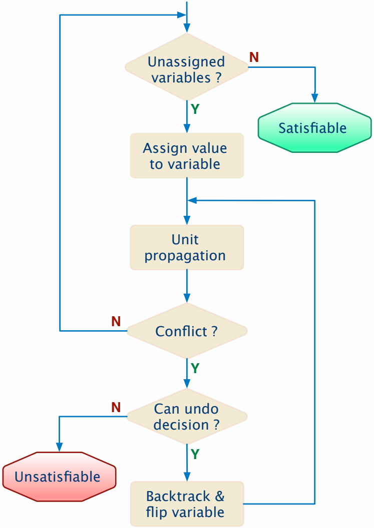

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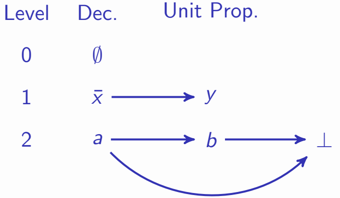

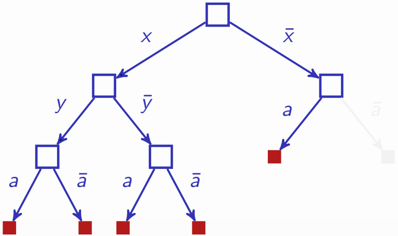

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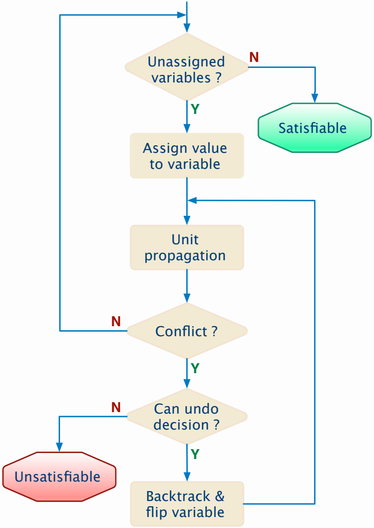

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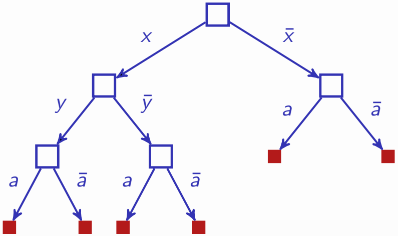

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How significant is CDCL SAT solving?

- Sample of solvers:
  1. POSIT: state of the art DPLL SAT solver in 1995
  2. GRASP: first CDCL SAT solver, state of the art 1995∼2000
  3. Minisat: CDCL SAT solver, state of the art until the late 00s
  4. Glucose: modern state of the art CDCL SAT solver
  5. ...

  – **Demo 1**: model checking example (from IBM)

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  – **Demo 1**: model checking example (from IBM)

- **Demo 2**: cooperative path finding (CPF)

What is a CDCL SAT solver?

- Extend DPLL SAT solver with: [DP60,DLL62]
  - Clause learning & non-chronological backtracking [MSS96a,MSS99,BS97,Z97]
  - Search restarts [GSK98,BMS00,H07,B08]
  - Lazy data structures
  - Conflict-guided branching
  - ...

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- Extend DPLL SAT solver with: [DP60,DLL62]
  - Clause learning & non-chronological backtracking [MSS96a,MSS99,BS97,Z97]
    - Exploit UIPs [MSS96a,SSS12]
    - Minimize learned clauses [SB09,VG09]
    - Opportunistically delete clauses [MSS96a,MSS99,GN02]
  - Search restarts [GSK98,BMS00,H07,B08]
  - Lazy data structures
    - Watched literals [MMZZM01]
  - Conflict-guided branching
    - Lightweight branching heuristics [MMZZM01]
    - Phase saving [S00,PD07]
  - ...

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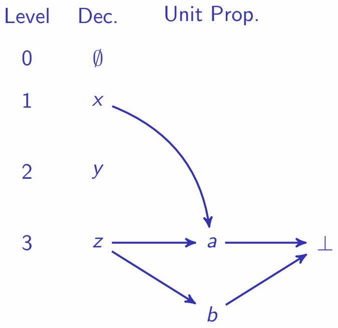
Clause Learning, UIPs & Minimization

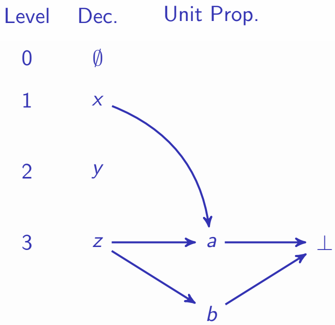
Search Restarts & Lazy Data Structures

Why CDCL Works?

Clause learning



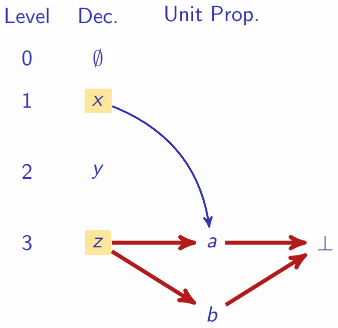
Clause learning



- Analyze conflict

[MSS96a,MSS96b,MSS96c,MSS96d,MSS99]

Clause learning



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    - Decision variable & literals assigned at decision levels less than current

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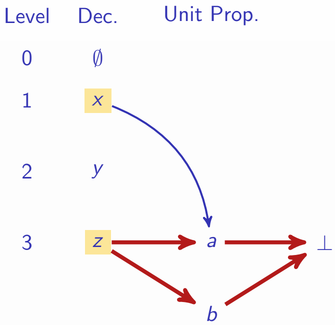

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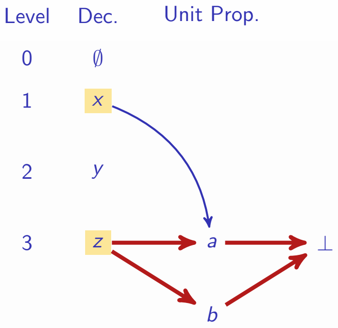
  - Create **new** clause: $(\bar{x} \vee \bar{z})$

Clause learning



$(\bar{a} \vee \bar{b})$ $(\bar{z} \vee b)$ $(\bar{x} \vee \bar{z} \vee a)$

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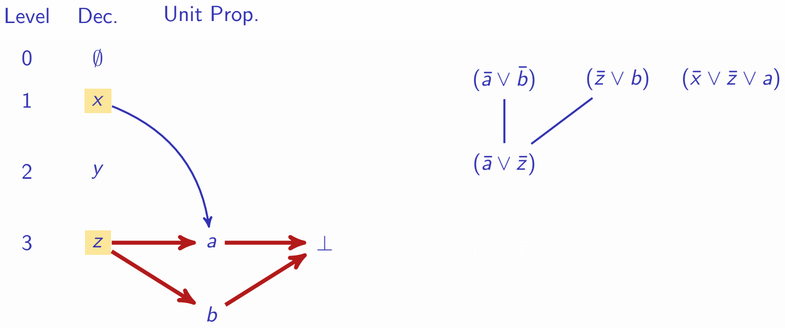
- Can relate clause learning with resolution

Clause learning



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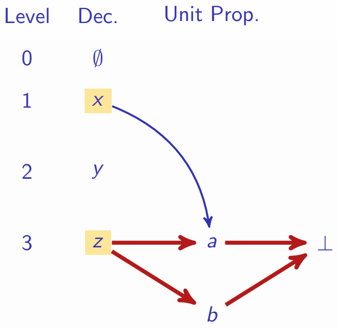
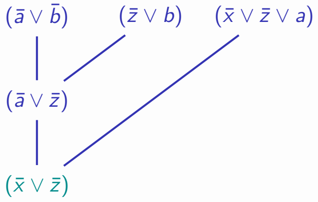
Clause learning





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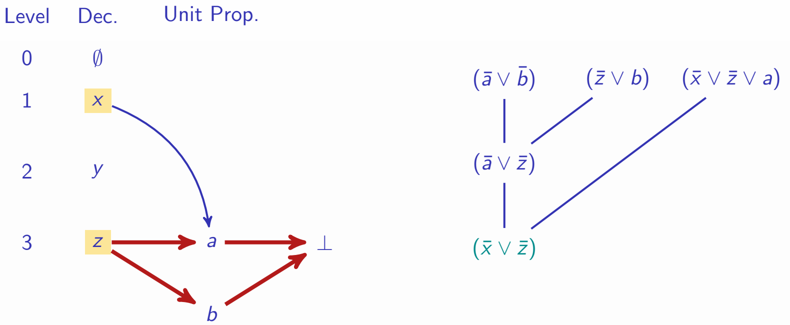

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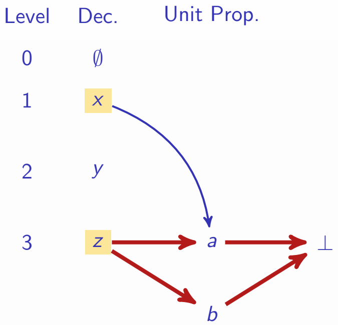

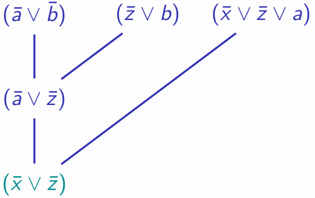

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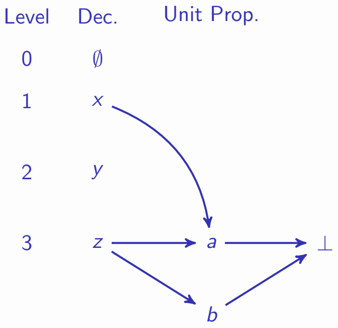
- **Note: GRASP-like clause learning**
  - Other instantiations of clause learning exist

Clause learning – after backtracking



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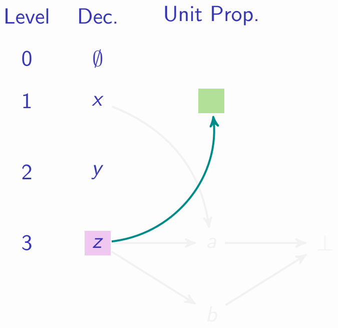

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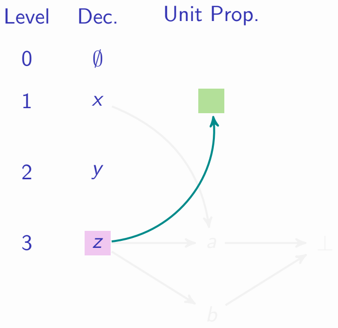

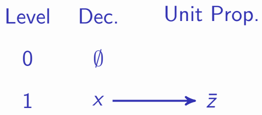

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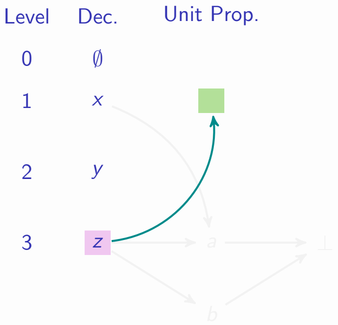
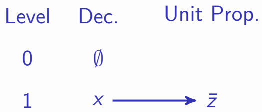
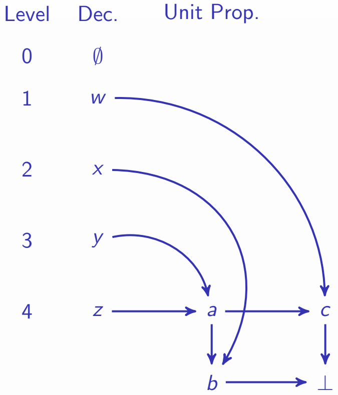




- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are asserting (with exceptions) [MSS96a,MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [ZMMM01]

Unique implication points (UIPs)



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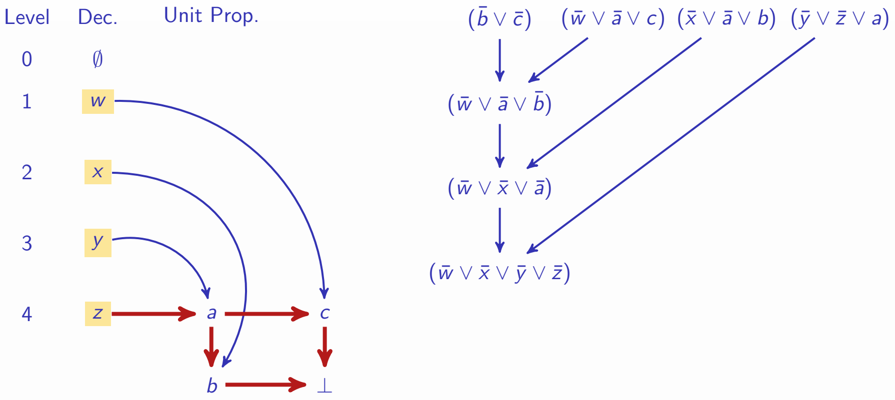

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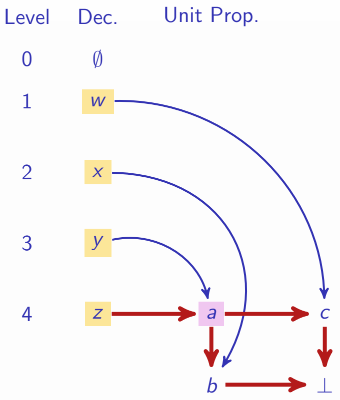

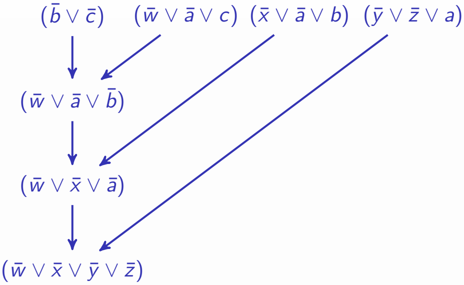

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[MSS96a,MSS99]

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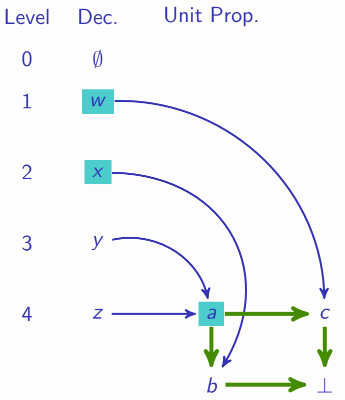

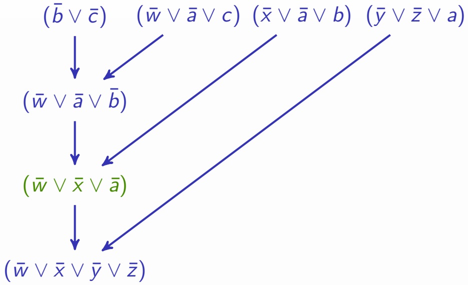

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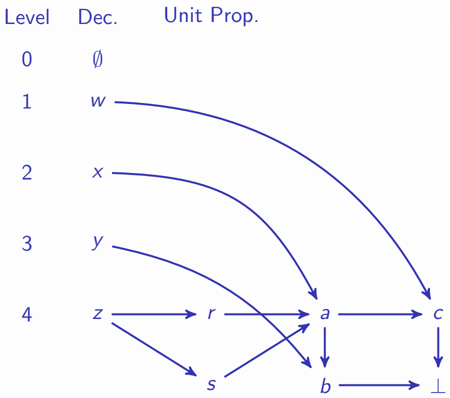

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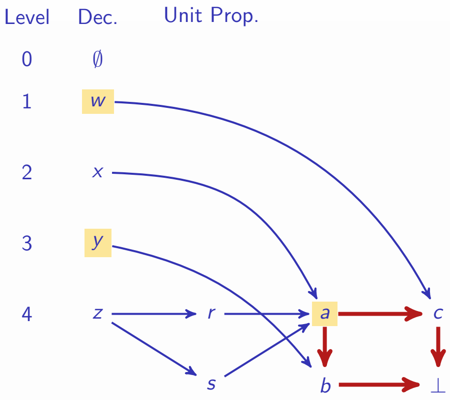

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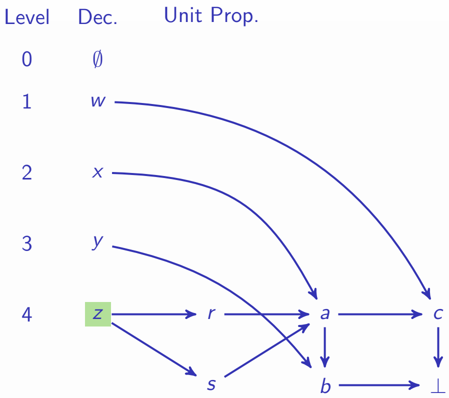

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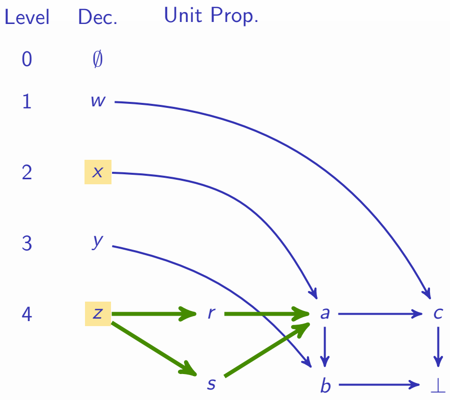

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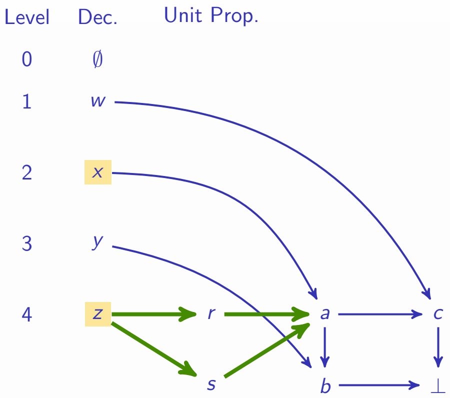

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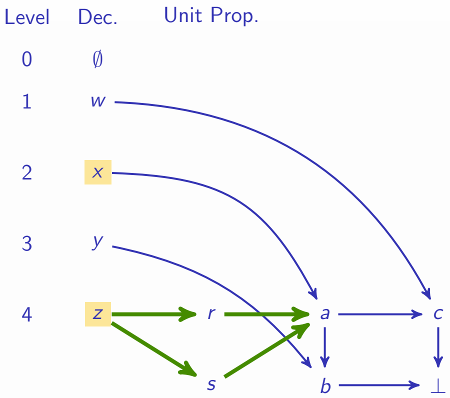

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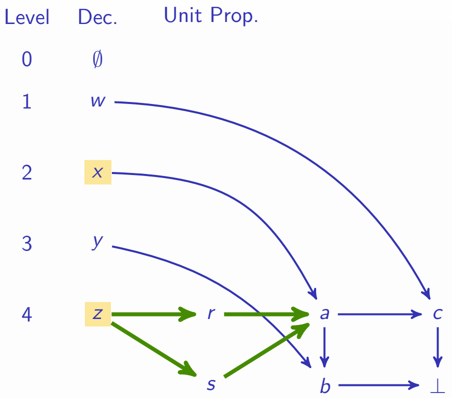

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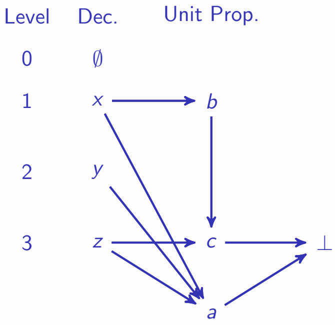
- Recent results show it can be beneficial on some instances [SSS12]

Clause minimization I



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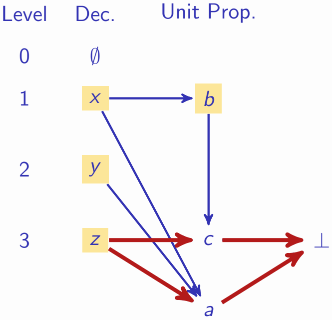

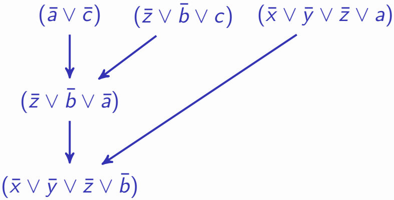

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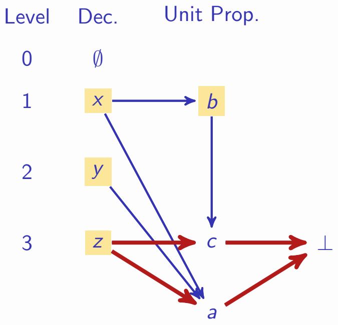

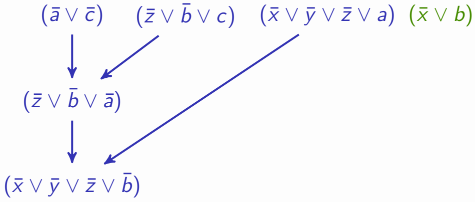

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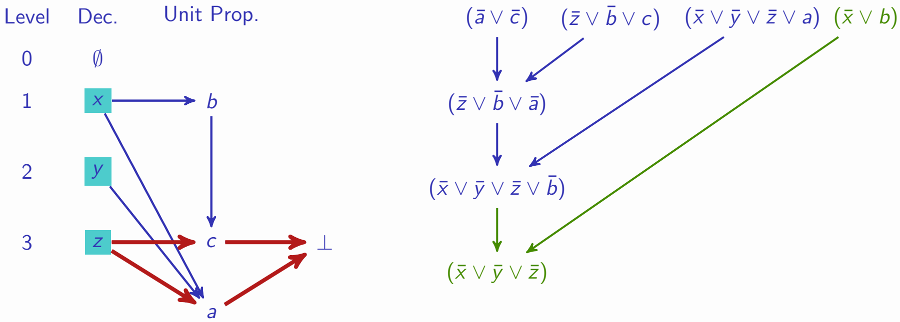

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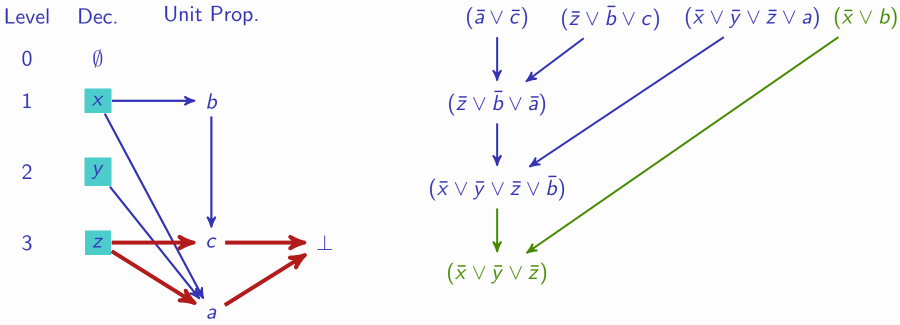

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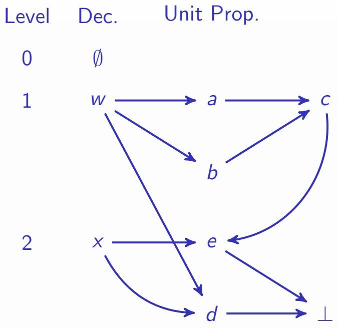

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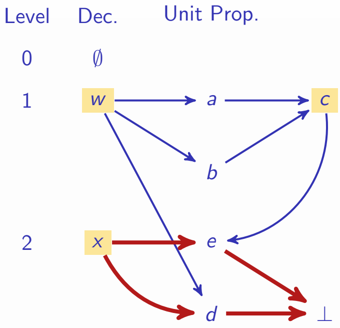

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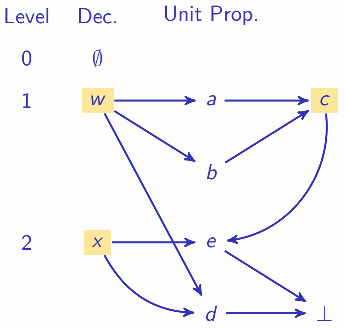

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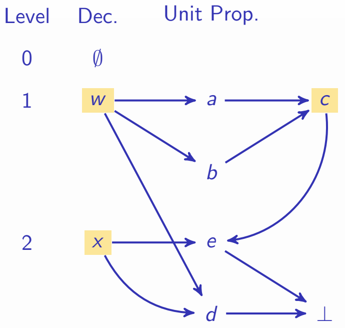

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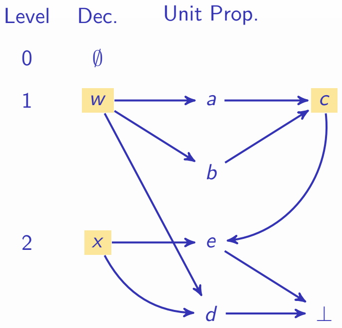

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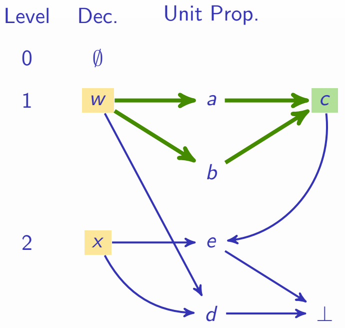

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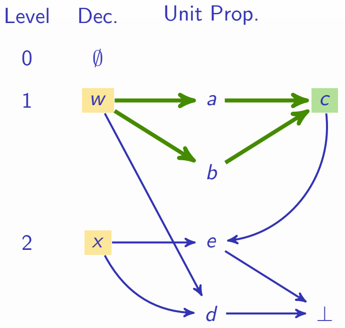

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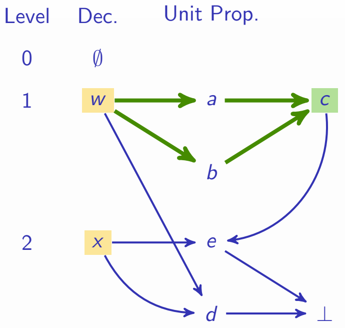

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- Recursive minimization runs in (amortized) linear time

Outline

Basic Definitions

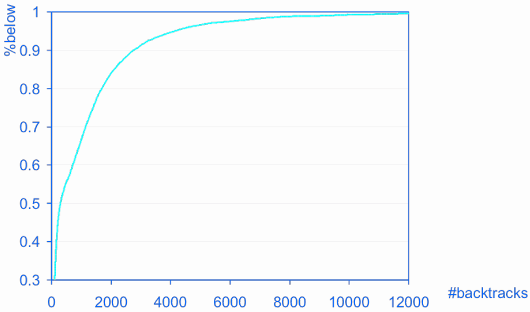
Clause Learning, UIPs & Minimization

Search Restarts & Lazy Data Structures

Why CDCL Works?

Search restarts I

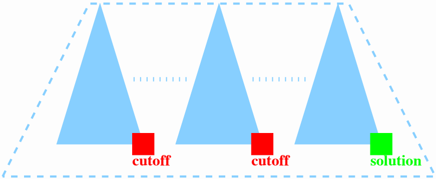
- Heavy-tail behavior: [GSK98]



  - 10000 runs, branching randomization on satisfiable industrial instance
    - Use rapid randomized restarts (search restarts)

Search restarts II

- Restart search after a number of conflicts



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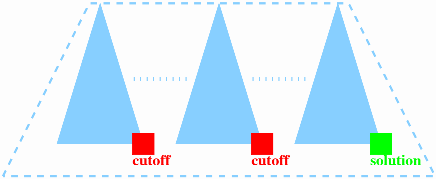

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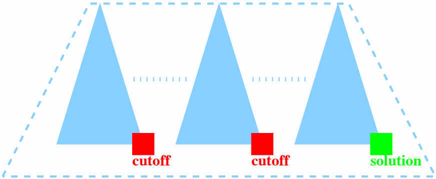

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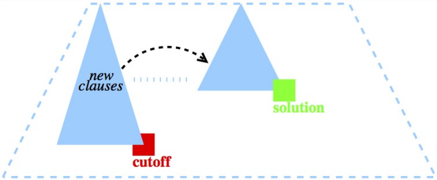
  - But there exist proof complexity arguments
- Learned clauses effective after restart(s)



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  - Clause learning can generate **large** clauses
    - Worst-case size: $\mathcal{O}(n)$
  - Worst-case number of literals: $\mathcal{O}(m\,n)$
  - In practice,

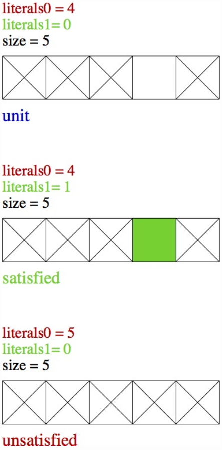
    Unit propagation slow-down worse than linear as clauses are learned **!**

- Clause learning to be effective requires a more efficient representation: **Watched Literals** [MMZZM01]
  - Watched literals are one example of lazy data structures
    - But there are others

Watched literals

[MMZZM01]

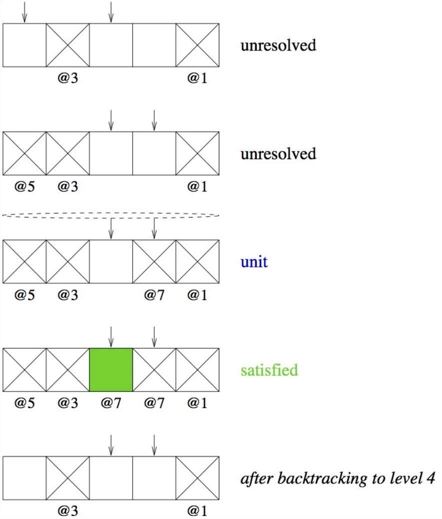
- Important states of a clause



Watched literals

[MMZZM01]

- Important states of a clause
- Associate **2** references with each clause



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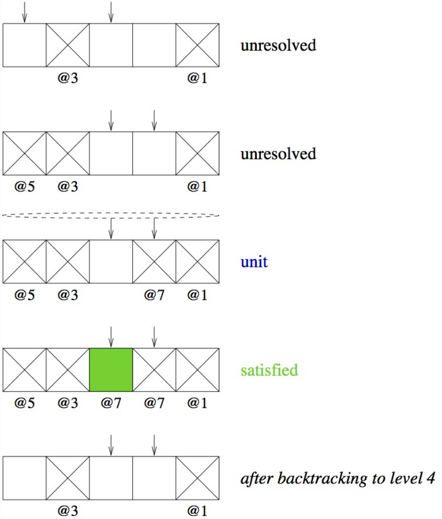
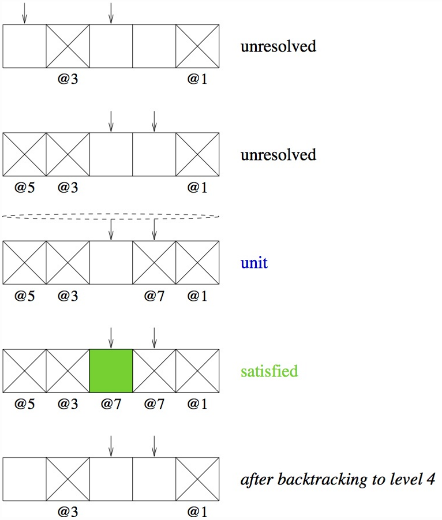


Watched literals

[MMZZM01]

- Important states of a clause
- Associate **2** references with each clause
- Deciding unit requires traversing all literals
- References **unchanged** when backtracking



Additional key techniques

- Lightweight branching [MMZZM01]
  - Use conflict to bias variables to branch on, associate score with each variable
  - Prefer recent bias by regularly decreasing variable scores
  - Recent promising ML-based branching [LGPC16a,LGPC16b]

Additional key techniques

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  - Not practical to keep all learned clauses
  - Delete larger clauses [E.g. MSS96a,MSS99]
  - Delete less used clauses [E.g. GN02,ES03]

Additional key techniques

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- Clause deletion policies
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  - Delete larger clauses [E.g. MSS96a,MSS99]
  - Delete less used clauses [E.g. GN02,ES03]
- Other effective techniques:
  - Phase saving [S00,PD07]
  - Luby restarts [H07]
  - Literal blocks distance [AS09]
  - Preprocessing/inprocessing [E.g. JHB12,HJLSB15]

Outline

Basic Definitions

Clause Learning, UIPs & Minimization

Search Restarts & Lazy Data Structures

Why CDCL Works?

Why CDCL works – a practitioner's view

- GRASP-like clause learning extensively inspired in circuit reasoners
  - UIPs mimic unique sensitization points (USPs), from testing
  - Analysis of conflicts organized by decision levels
    - In circuits, branching is (mostly) on the inputs, e.g. PODEM, FAN, etc.
    - Need to find ways to exploit the circuit's internal structure
    - Several ideas originated in earlier work [MSS94a,MSS94b]

- There are also proof complexity arguments

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    - Need to find ways to exploit the circuit's internal structure
    - Several ideas originated in earlier work [MSS94a,MSS94b]
- Understanding problem structure is **essential**
  - Clauses are learned locally to each decision level
  - UIPs further localize the learned clauses
  - GRASP-like clause learning aims at learning small clauses, related with the sources of conflicts
  - Most practical problem instances exhibit the structure GRASP-like clause learning is most effective on
    - Most problems are **not** natively represented in clausal form [S13]
- There are also proof complexity arguments

Part II

Problem Modeling for SAT

Outline

Recap Clausification of Boolean Formulas

Hard and Soft Constraints

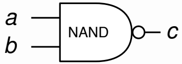
Linear Constraints

Encoding CSPs

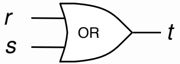
Modeling Examples & Exercises

- Satisfiability problems can be defined on Boolean circuits/formulas
  - Can use any logic connective: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots$
- Can represent circuits/formulas as CNF formulas [T68,PG86]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate's inputs and output
    - Given $z = \mathrm{OP}(x, y)$, represent in CNF $z \leftrightarrow \mathrm{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

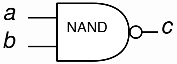
$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$



$$\mathcal{F}_t = (\bar{r} \vee t) \wedge (\bar{s} \vee t) \wedge (r \vee s \vee \bar{t})$$



Representing Boolean formulas / circuits II



c \ ab	00	01	11	10
0	0	0	1	0
1	1	1	0	1

a	b	c	$\mathcal{F}_c$(a,b,c)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
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1	1	0	1
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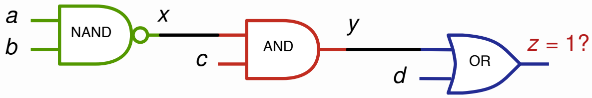
$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

Representing Boolean formulas / circuits III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



$$
\begin{aligned}
\mathcal{F} \quad = \quad & (a \vee x) \wedge (b \vee x) \wedge (\bar{a} \vee \bar{b} \vee \bar{x}) \wedge \\
& (x \vee \bar{y}) \wedge (c \vee \bar{y}) \wedge (\bar{x} \vee \bar{c} \vee y) \wedge \\
& (\bar{y} \vee z) \wedge (\bar{d} \vee z) \wedge (y \vee d \vee \bar{z}) \wedge (z)
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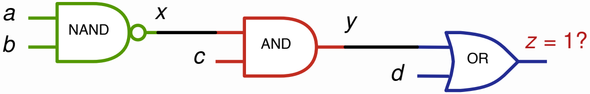

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- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and (non-clausal) formulas, besides adding new variables

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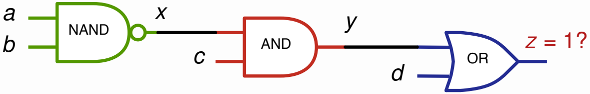

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- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and (non-clausal) formulas, besides adding new variables
- Easy to do more structures: ITEs; Adders; etc.

Outline

Recap Clausification of Boolean Formulas

Hard and Soft Constraints

Linear Constraints

Encoding CSPs

Modeling Examples & Exercises

Hard vs. soft constraints

- **Hard**: Constraints that **must** be satisfied

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  - How to model linear cost function optimization?

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  - Hard constraints: φ
  - Soft constraints: (x_j), each with cost c_j

Outline

Recap Clausification of Boolean Formulas

Hard and Soft Constraints

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Modeling Examples & Exercises

Linear constraints

- Cardinality constraints: $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

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- Pseudo-Boolean constraints: $\sum_{j=1}^{n} a_j x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

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- Pseudo-Boolean constraints: $\sum_{j=1}^{n} a_j x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

- If variables are non-Boolean, e.g. with finite domain
  - Need to encode variables (more later)

Equals1, AtLeast1 & AtMost1 constraints

- $\sum_{j=1}^{n} x_j = 1$: encode with $(\sum_{j=1}^{n} x_j \leq 1) \wedge (\sum_{j=1}^{n} x_j \geq 1)$

- $\sum_{j=1}^{n} x_j \geq 1$: encode with $(x_1 \vee x_2 \vee \ldots \vee x_n)$

- $\sum_{j=1}^{n} x_j \leq 1$ encode with:
  - Pairwise encoding
    - Clauses: $\mathcal{O}(n^2)$; No auxiliary variables
  - Sequential counter [S05]
    - Clauses: $\mathcal{O}(n)$; Auxiliary variables: $\mathcal{O}(n)$
  - Bitwise encoding [P07,FP01]
    - Clauses: $\mathcal{O}(n \log n)$; Auxiliary variables: $\mathcal{O}(\log n)$
  - ...

Pairwise encoding

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$$
\begin{aligned}
a \rightarrow \bar{b} \wedge \bar{c} \wedge \bar{d} \quad &\Longrightarrow \quad (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}) \wedge (\bar{a} \vee \bar{d}) \\
b \rightarrow \bar{c} \wedge \bar{d} \wedge \bar{a} \quad &\Longrightarrow \quad (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{b} \vee \bar{a}) \\
c \rightarrow \bar{d} \wedge \bar{a} \wedge \bar{b} \quad &\Longrightarrow \quad (\bar{c} \vee \bar{d}) \wedge (\bar{c} \vee \bar{a}) \wedge (\bar{c} \vee \bar{b}) \\
d \rightarrow \bar{a} \wedge \bar{b} \wedge \bar{c} \quad &\Longrightarrow \quad (\bar{d} \vee \bar{a}) \wedge (\bar{d} \vee \bar{b}) \wedge (\bar{d} \vee \bar{c})
\end{aligned}
$$

  - Encoded as: $(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}) \wedge (\bar{a} \vee \bar{d}) \wedge (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{c} \vee \bar{d})$

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- With N variables, number of clauses becomes $\frac{n(n-1)}{2}$
  - But no additional variables

Sequential counter encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with sequential counter:

$$(\bar{x}_1 \vee s_1) \wedge (\bar{x}_n \vee \bar{s}_{n-1}) \wedge$$
$$\bigwedge_{1<i<n} \left((\bar{x}_i \vee s_i) \wedge (\bar{s}_{i-1} \vee s_i) \wedge (\bar{x}_i \vee \bar{s}_{i-1}) \right)$$

  - If some $x_j = 1$, then all s_i variables must be assigned
    - $s_i = 1$ for $i \geq j$, and so $x_i = 0$ for $i > j$
    - $s_i = 0$ for $i < j$, and so $x_i = 0$ for $i < j$
    - Thus, all other x_i variables must take value 0
  - If all $x_j = 0$, can find consistent assignment to s_i variables
  - $\mathcal{O}(n)$ clauses ; $\mathcal{O}(n)$ auxiliary variables

Bitwise encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with bitwise encoding:

- An example: $x_1 + x_2 + x_3 \leq 1$

Bitwise encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with bitwise encoding:
  - Auxiliary variables $v_0, \ldots, v_{r-1}$; $r = \lceil \log n \rceil$ (with $n > 1$)
  - If $x_j = 1$, then $v_0 \ldots v_{r-1} = b_0 \ldots b_{r-1}$, the binary encoding of $j - 1$

    $x_j \rightarrow (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}) \Leftrightarrow (\bar{x}_j \vee (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}))$

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
x_1	0	00
x_2	1	01
x_3	2	10

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  - Clauses $(\bar{x}_j \vee (v_i \leftrightarrow b_i)) = (\bar{x}_j \vee l_i)$, $i = 0, \ldots, r-1$, where
    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
x_1	0	00
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$(\bar{x}_1 \vee \bar{v}_1) \wedge (\bar{x}_1 \vee \bar{v}_0)$
$(\bar{x}_2 \vee \bar{v}_1) \wedge (\bar{x}_2 \vee v_0)$
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    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise
  - If $x_j = 1$, assignment to v_i variables must encode $j - 1$
    - For consistency, all other x variables must not take value 1
  - If all $x_j = 0$, any assignment to v_i variables is consistent
  - $\mathcal{O}(n \log n)$ clauses ; $\mathcal{O}(\log n)$ auxiliary variables

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
x_1	0	00
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$$(\bar{x}_1 \vee \bar{v}_1) \wedge (\bar{x}_1 \vee \bar{v}_0)$$
$$(\bar{x}_2 \vee \bar{v}_1) \wedge (\bar{x}_2 \vee v_0)$$
$$(\bar{x}_3 \vee v_1) \wedge (\bar{x}_3 \vee \bar{v}_0)$$

General cardinality constraints

- General form: $\sum_{j=1}^{n} x_j \leq k$ (or $\sum_{j=1}^{n} x_j \geq k$)
  - Operational encoding [W98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not guarantee arc-consistency
  - Generalized pairwise
    - Clauses: $\mathcal{O}(2^n)$; no auxiliary variables
  - Sequential counters [S05]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - BDDs [ES06]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - Sorting networks [ES06]
    - Clauses/Variables: $\mathcal{O}(n \log^2 n)$
  - Cardinality Networks: [ANORC09,ANORC11a]
    - Clauses/Variables: $\mathcal{O}(n \log^2 k)$
  - Pairwise Cardinality Networks: [CZI10]
  - ...

Generalized pairwise encoding

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$$\begin{array}{lcl} a \wedge b \rightarrow \bar{c} & \Longrightarrow & (\bar{a} \vee \bar{b} \vee \bar{c}) \\ a \wedge b \rightarrow \bar{d} & \Longrightarrow & (\bar{a} \vee \bar{b} \vee \bar{d}) \\ a \wedge c \rightarrow \bar{d} & \Longrightarrow & (\bar{a} \vee \bar{c} \vee \bar{d}) \\ b \wedge c \rightarrow \bar{d} & \Longrightarrow & (\bar{b} \vee \bar{c} \vee \bar{d}) \end{array}$$

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Generalized pairwise encoding

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a \wedge c \rightarrow \bar{d} & \Longrightarrow & (\bar{a} \vee \bar{c} \vee \bar{d}) \\
b \wedge c \rightarrow \bar{d} & \Longrightarrow & (\bar{b} \vee \bar{c} \vee \bar{d})
\end{array}
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- In general, number of clauses is C_{k+1}^{n}
  - Recall: for AtMost1 (i.e. for $k = 1$), number of clauses is: $\frac{n(n-1)}{2}$

Another example

- Example: $a + b + c + d + e \leq 2$
- Encoding will contain $C_3^5 = 10$ clauses:

$$
\begin{aligned}
a \wedge b \rightarrow \bar{c} &\implies (\bar{a} \vee \bar{b} \vee \bar{c}) \\
a \wedge b \rightarrow \bar{d} &\implies (\bar{a} \vee \bar{b} \vee \bar{d}) \\
a \wedge b \rightarrow \bar{e} &\implies (\bar{a} \vee \bar{b} \vee \bar{e}) \\
a \wedge c \rightarrow \bar{d} &\implies (\bar{a} \vee \bar{c} \vee \bar{d}) \\
a \wedge c \rightarrow \bar{e} &\implies (\bar{a} \vee \bar{c} \vee \bar{e}) \\
a \wedge d \rightarrow \bar{e} &\implies (\bar{a} \vee \bar{d} \vee \bar{e}) \\
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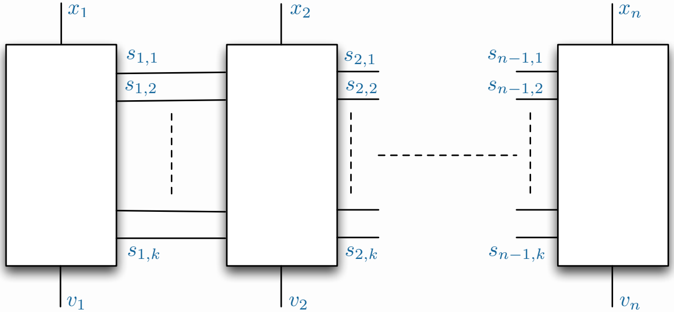
c \wedge d \rightarrow \bar{e} &\implies (\bar{c} \vee \bar{d} \vee \bar{e})
\end{aligned}
$$

Sequential counter – revisited I

- Encode $\sum_{j=1}^{n} x_j \leq k$ with sequential counter:



- Equations for each block $1 < i < n$, $1 < j < k$:

$s_i = \sum_{j=1}^{i} x_j$

s_i represented in unary

$s_{i,1} = s_{i-1,1} \vee x_i$

$s_{i,j} = s_{i-1,j} \vee s_{i-1,j-1} \wedge x_i$

$v_i = (s_{i-1,k} \wedge x_i) = 0$

Sequential counter – revisited II

- CNF formula for $\sum_{j=1}^{n} x_j \leq k$:
  - Assume: $k > 0 \land n > 1$
  - Indeces: $1 < i < n \;,\; 1 < j \leq k$

$$
\begin{array}{l}
(\neg x_1 \lor x_{1,1}) \\
(\neg s_{1,j}) \\
(\neg x_i \lor s_{i,1}) \\
(\neg s_{i-1,1} \lor s_{i,1}) \\
(\neg x_i \lor \neg s_{i-1,j-1} \lor s_{i,j}) \\
(\neg s_{i-1,j} \lor s_{i,j}) \\
(\neg x_i \lor \neg s_{i-1,k}) \\
(\neg x_n \lor \neg s_{n-1,k})
\end{array}
$$

- $\mathcal{O}(n\ k)$ clauses & variables

Pseudo-Boolean constraints

- General form: $\sum_{j=1}^{n} a_j\, x_j \leq b$
  - Operational encoding [W98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not guarantee arc-consistency
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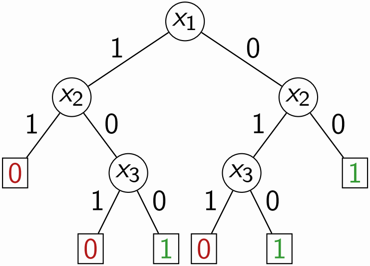
  - Improved polynomial watchdog encoding [ANORC11b]
    - Clauses & aux variables: $\mathcal{O}(n^3\log(a_{max}))$
  - ...

Encoding PB constraints with BDDs I

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Construct BDD
  - E.g. analyze variables by decreasing coefficients
- Extract ITE-based circuit from BDD



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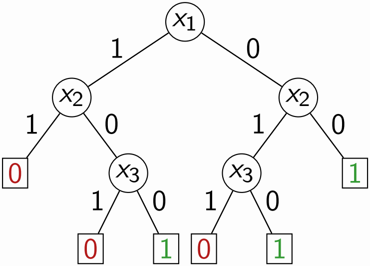
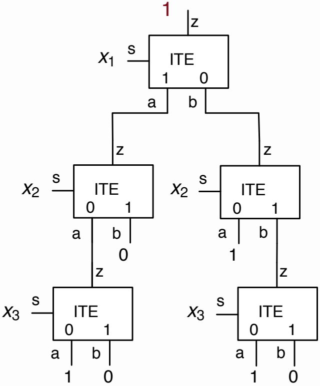
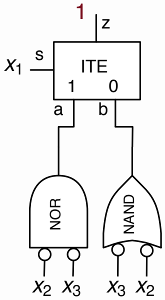




Encoding PB constraints with BDDs II

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Extract ITE-based circuit from BDD
- Simplify and create final circuit:



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  - Let $x_2 = 0$
  - Either constraint can still be satisfied, but **not** both

Outline

Recap Clausification of Boolean Formulas

Hard and Soft Constraints

Linear Constraints

Encoding CSPs

Modeling Examples & Exercises

CSP constraints

- Many possible encodings:
  - Direct encoding [dK89,GJ96,W00]
  - Log encoding [W00]
  - Support encoding [K90,G02]
  - Log-Support encoding [G07]
  - Order encoding for finite linear CSPs [TTKB09]

Direct encoding for CSP w/ binary constraints

- Variable x_i with domain D_i, with $m_i = |D_i|$
- Constraints are relations over domains of variables
  - For a constraint over $x_1, \ldots, x_k$, define relation $R \subseteq D_1 \times \cdots \times D_k$
  - Need to encode elements not in the relation
  - For a binary relation, use set of binary clauses, one for each element not in R
- Represent values of x_i with Boolean variables $x_{i,1}, \ldots, x_{i,m_i}$
- Require $\sum_{k=1}^{m_i} x_{i,k} = 1$
  - Suffices to require $\sum_{k=1}^{m_i} x_{i,k} \geq 1$ [W00]
- If the pair of assignments $x_i = v_i \wedge x_j = v_j$ is not allowed, add binary clause $(\bar{x}_{i,v_i} \vee \bar{x}_{j,v_j})$

Additional topics

- Encoding problems to SAT is ubiquitous:
  - Many more encodings of finite domain CSP into SAT
  - Encodings of Answer Set Programming (ASP) into SAT
  - Eager SMT solving
  - Theorem provers iteratively encode problems into SAT
  - Model finders interatively encode problems into SAT
  - ...

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Recap Clausification of Boolean Formulas

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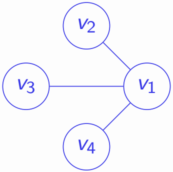
Modeling Examples & Exercises

Minimum vertex cover

- The problem:
  - Graph $G = (V, E)$
  - Vertex cover $U \subseteq V$
    - For each $(v_i, v_j) \in E$, either $v_i \in U$ or $v_j \in U$
  - Minimum vertex cover: vertex cover U of minimum size



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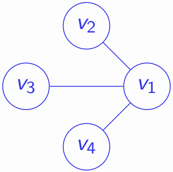

Vertex cover: $\{v_2, v_3, v_4\}$

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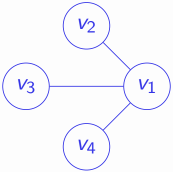

Vertex cover: $\{v_2, v_3, v_4\}$

Min vertex cover: $\{v_1\}$

Minimum vertex cover

- Modeling with Pseudo-Boolean Optimization (PBO):
  - Variables: x_i for each $v_i \in V$, with $x_i = 1$ iff $v_i \in U$
  - Clauses: $(x_i \vee x_j)$ for each $(v_i, v_j) \in E$
  - Objective function: minimize number of true x_i variables
    - I.e. minimize vertices included in U

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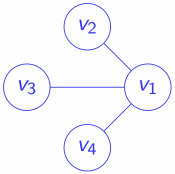

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} & (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_1 \vee x_4) \end{array}$$

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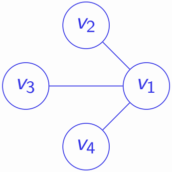

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\end{array}
$$

- Alternative propositional encoding:

$$
\begin{aligned}
\varphi_S &= \{(\neg x_1), (\neg x_2), (\neg x_3), (\neg x_4)\} \\
\varphi_H &= \{(x_1 \vee x_2), (x_1 \vee x_3), (x_1 \vee x_4)\}
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Graph coloring

- Given undirected graph $G = (V, E)$ and k colors:
  - Can we assign colors to vertices of G s.t. any pair of adjacent vertices are assigned different colors?

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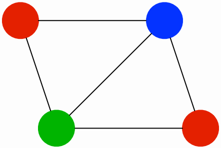
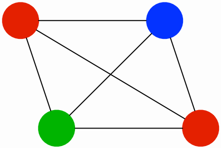


Valid coloring



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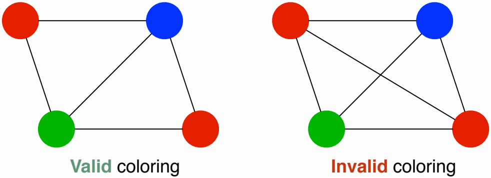

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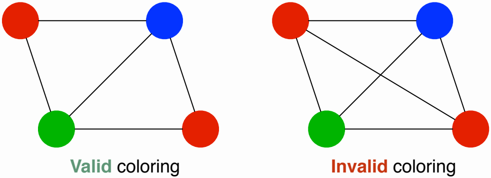

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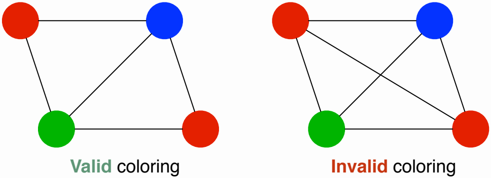

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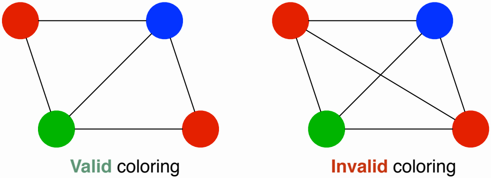

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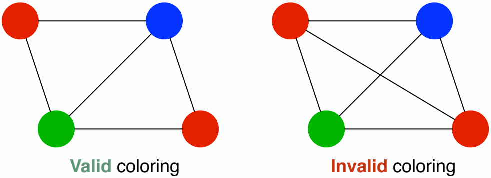

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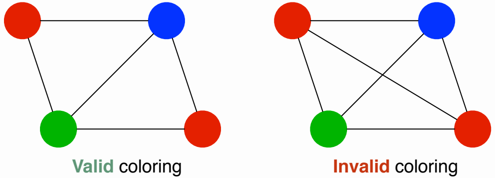

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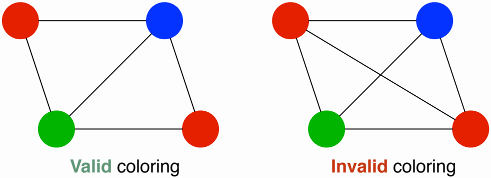

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  - **Note: it suffices to use** $\left(\bigvee_{j \in \{1,\ldots,k\}} x_{i,j}\right)$

The N-Queens problem I

- The N-Queens Problem:
  Place N queens on a $N \times N$ board, such that no two queens attack each other

- Example for a 5×5 board:

Q				
			Q	
	Q			
				Q
		Q		

The N-Queens problem II

- x_{ij}: 1 if queen placed in position (i, j); 0 otherwise
- Each row must have exactly one queen:

$$1 \leq i \leq N, \qquad \sum_{j=1}^{N} x_{ij} = 1$$

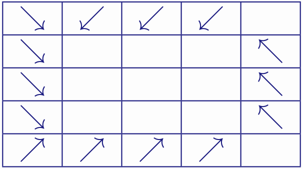
- Each column must have exactly one queen:

$$1 \leq j \leq N, \qquad \sum_{i=1}^{N} x_{ij} = 1$$

- Also, need to define constraints on diagonals...

The N-Queens problem III

- Each diagonal can have at most one queen:



$$i = 1, \quad 2 \leq j < N, \quad \sum_{k=0}^{j-1} x_{i+k\, j-k} \leq 1$$

$$i = N, \quad 1 \leq j < N, \quad \sum_{k=0}^{N-j} x_{i-k\, j+k} \leq 1$$

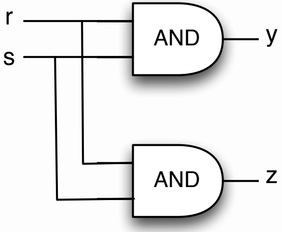
$$j = 1, \quad 1 \leq i < N, \quad \sum_{k=0}^{N-i} x_{i+k\, j+k} \leq 1$$

$$j = N, \quad 2 \leq i < N, \quad \sum_{k=0}^{i-1} x_{i-k\, j-k} \leq 1$$

Design debugging

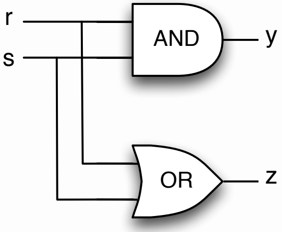
[SMVLS'07]

Correct circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Valid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

Faulty circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Invalid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

- The model:
  - Hard clauses: Input and output values
  - Soft clauses: CNF representation of circuit
- The problem:
  - Maximize number of satisfied clauses (i.e. circuit gates)

Software package upgrades

[MBCV'06,TSJL'07,AL'08,ALMS'09,ALBL'10]

- Universe of software packages: $\{p_1, \ldots, p_n\}$
- Associate x_i with p_i: $x_i = 1$ iff p_i is installed
- Constraints associated with package p_i: (p_i, D_i, C_i)
  - D_i: dependencies (required packages) for installing p_i
  - C_i: conflicts (disallowed packages) for installing p_i
- Example problem: Maximum Installability
  - Maximum number of packages that can be installed
  - Package constraints represent hard clauses
  - Soft clauses: (x_i)

Package constraints:

$(p_1, \{p_2 \vee p_3\}, \{p_4\})$
$(p_2, \{p_3\}, \{p_4\})$
$(p_3, \{p_2\}, \emptyset)$
$(p_4, \{p_2, p_3\}, \emptyset)$

Software package upgrades

[MBCV'06,TSJL'07,AL'08,ALMS'09,ALBL'10]

- Universe of software packages: $\{p_1, \ldots, p_n\}$
- Associate x_i with p_i: $x_i = 1$ iff p_i is installed
- Constraints associated with package p_i: (p_i, D_i, C_i)
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MaxSAT formulation:

$$\begin{aligned}\varphi_H &= \{(\neg x_1 \vee x_2 \vee x_3), (\neg x_1 \vee \neg x_4), \\ &\quad (\neg x_2 \vee x_3), (\neg x_2 \vee \neg x_4), (\neg x_3 \vee x_2), \\ &\quad (\neg x_4 \vee x_2), (\neg x_4 \vee x_3)\} \\ \varphi_S &= \{(x_1), (x_2), (x_3), (x_4)\}\end{aligned}$$

Exercise: knapsack

- Given list of pairs (v_i, w_i), $i = 1, \ldots, n$
  - Each pair (v_i, w_i), represents the value and weight of object i

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- Propositional encoding for the knapsack problem?

- **Hint:** consider 0-1 ILP (or PBO) formulation:
  - Associate propositional variable x_i with each objet i
  - $x_i = 1$ iff object i is picked

$$\begin{array}{ll} \text{max} & \sum_{i=1}^{n} v_i \cdot x_i \\ \text{s.t} & \sum_{i=1}^{n} w_i \cdot x_i \leq W \end{array}$$

Exercise: solving Sudoku I

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

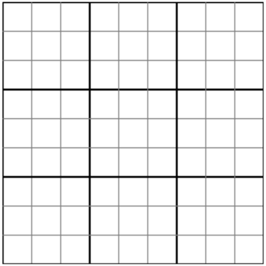
Exercise: solving Sudoku II

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Exercise: solving Sudoku II

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

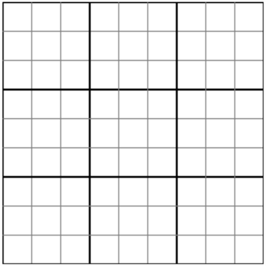
- How to solve Sudoku with SAT?



- Modeling the problem with integer variables:
  - Rows: $i = 1, \ldots, 9$
  - Columns: $j = 1, \ldots, 9$
  - Variables: $v_{i,j} \in \{1, 2, \ldots, 9\}$, $i, j \in \{1, \ldots, 9\}$

- Constraints:

Solving Sudoku – with constraints



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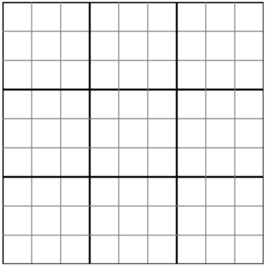
- Constraints:
  - Each value used exactly once in each row:
    - For $i \in \{1, \ldots, 9\}$: $\text{alldifferent}(v_{i,1}, \ldots, v_{i,9})$

Solving Sudoku – with constraints



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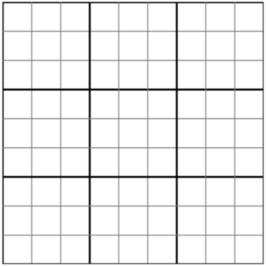
  - Each value used exactly once in each column:
    - For $j \in \{1, \ldots, 9\}$: $\text{alldifferent}(v_{1,j}, \ldots, v_{9,j})$

Solving Sudoku – with constraints



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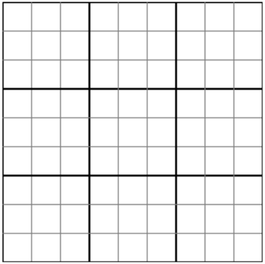
  - Each value used exactly once in each column:
    - For $j \in \{1, \ldots, 9\}$: $\text{alldifferent}(v_{1,j}, \ldots, v_{9,j})$
  - Each value used exactly once in each 3×3 sub-grid:
    - For $i, j \in \{0, 1, 2\}$:
      $\text{alldifferent}(v_{3i+1,3j+1}, v_{3i+1,3j+2}, v_{3i+1,3j+3}, v_{3i+2,3j+1}, \ldots, v_{3i+3,3j+1}, \ldots)$

Solving Sudoku – propositional logic – variables



- Modeling with propositional variables:
  - Rows: $i = 1, \dots, 9$
  - Columns: $j = 1, \dots, 9$
  - Variables: $v_{i,j,k} \in \{0, 1\}$, $i, j, k \in \{1, \dots, 9\}$

Solving Sudoku – propositional logic – constraints

- Value in each cell is valid:
  - For $i, j \in \{1, \ldots, 9\}$:

$$\sum_{k=1}^{9} v_{i,j,k} = 1$$

- Each value used exactly once in each row:
  - For $i \in \{1, \ldots, 9\}$, $k \in \{1, \ldots, 9\}$:

$$\sum_{j=1}^{9} v_{i,j,k} = 1$$

- Each value used exactly once in each column:
  - For $j \in \{1, \ldots, 9\}$, $k \in \{1, \ldots, 9\}$:

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- Each value used exactly once in each 3×3 sub-grid:
  - For $i, j \in \{0, 1, 2\}$, $k \in \{1, \ldots, 9\}$:

$$\sum_{r=1}^{3} \sum_{s=1}^{3} v_{3i+r,3j+s,k} = 1$$

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- **Q**: how to encode Equals1 constraints?

Constraints for fixed cells

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
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- Integer variables:

$$v_{1,1} = 5, v_{1,2} = 3, v_{1,5} = 7, v_{2,1} = 6, v_{2,4} = 1, v_{2,5} = 9$$
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- Propositional variables:

$$v_{1,1,5} = 1, v_{1,2,3} = 1, v_{1,5,7} = 1, v_{2,1,6} = 1, v_{2,4,1} = 1, v_{2,5,9} = 1$$
$$v_{2,6,5} = 1, v_{3,2,9} = 1, v_{3,3,8} = 1, v_{3,8,6} = 1, v_{4,1,8} = 1, v_{4,5,6} = 1, \ldots$$

Part III

Problem Solving with SAT Oracles

Computing a model

- **Q:** How to solve the FSAT problem?

  FSAT: Compute a model of a satisfiable CNF formula $\mathcal{F}$, using an NP oracle

Computing a model

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  - A possible algorithm:
    - Analyze each variable $x_i \in \{x_1, \ldots, x_n\} = \text{var}(\mathcal{F})$
    - Consider $\mathcal{F} \wedge (x_i)$. Call NP oracle. If answer is **yes**, then add (x_i) to $\mathcal{F}$. If answer is **no**, then add $(\neg x_i)$ to $\mathcal{F}$

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  - **Note:** Cannot solve FSAT with logarithmic number of NP oracle calls, unless P = NP [GF93]

- FSAT is an example of a function problem

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- FSAT is an example of a function problem
  - **Note:** FSAT can be solved with **one** SAT oracle call

Beyond decision problems

Answer	Problem Type

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems

Beyond decision problems

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Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	Enumeration Problems

Beyond decision problems

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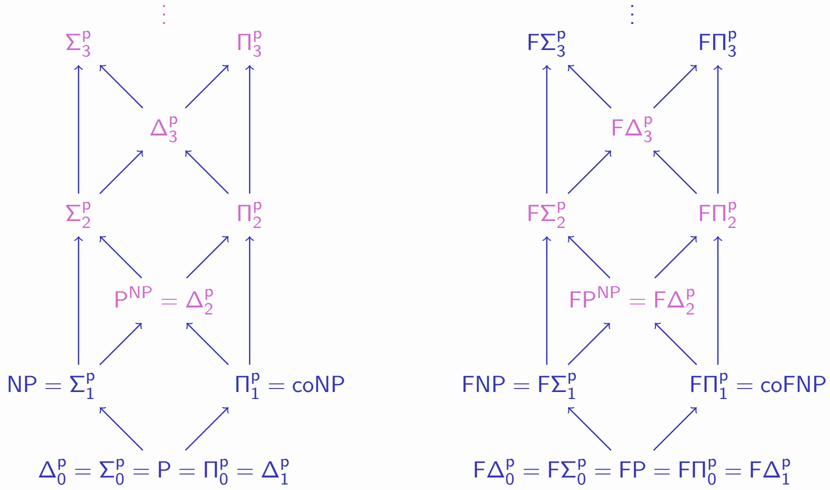
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	Enumeration Problems
# solutions	

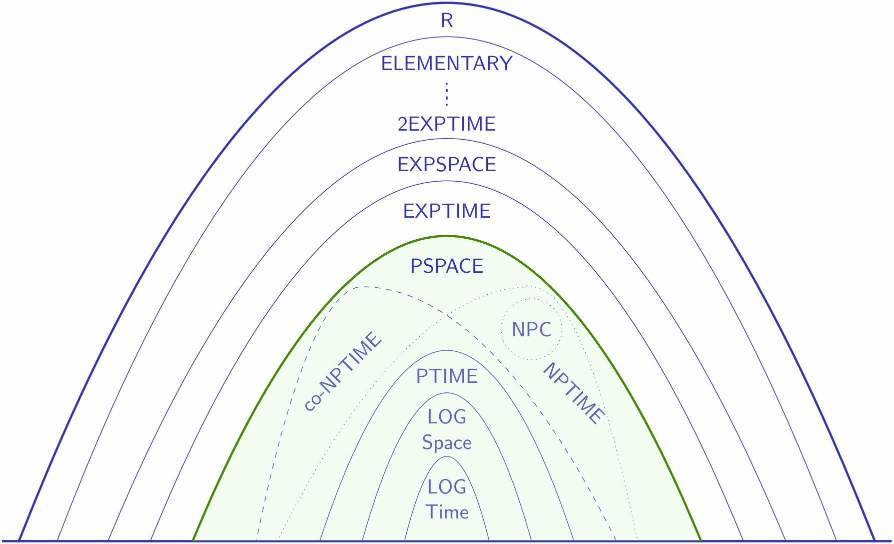
Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	Enumeration Problems
# solutions	Counting Problems

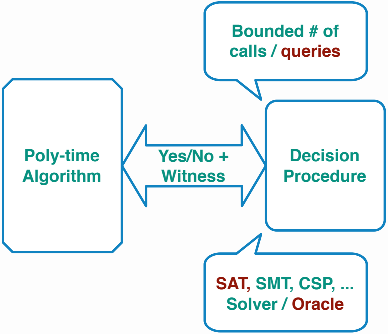
... and beyond NP – decision and function problems



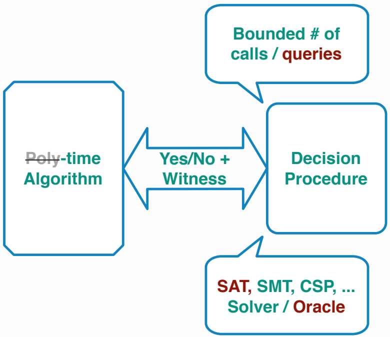
... and beyond NP – our current range



Oracle-based problem solving – ideal scenario



Oracle-based problem solving – in some settings



Many problems to solve – within FP^{NP}

Answer	Problem Type
Yes/No	Decision Problems
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Function Problems on Propositional Formulas

MaxSAT **PBO** **...** **WBO** **MinSAT**

Minimal Models **Prime Implicants** **Maximal Models** **Autarkies** **Backbones** **...** **Prime Implicates** **MUSes** **MCSes** **MESes** **Indep. Vars** **MDSes** **Implicant Ext.** **MFSes** **MSSes** **MNSes** **Implicate Ext.** **MCFSes**

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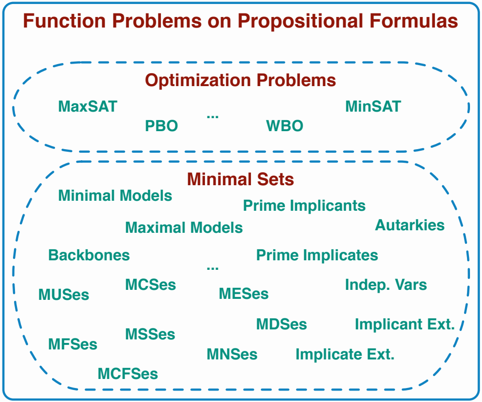
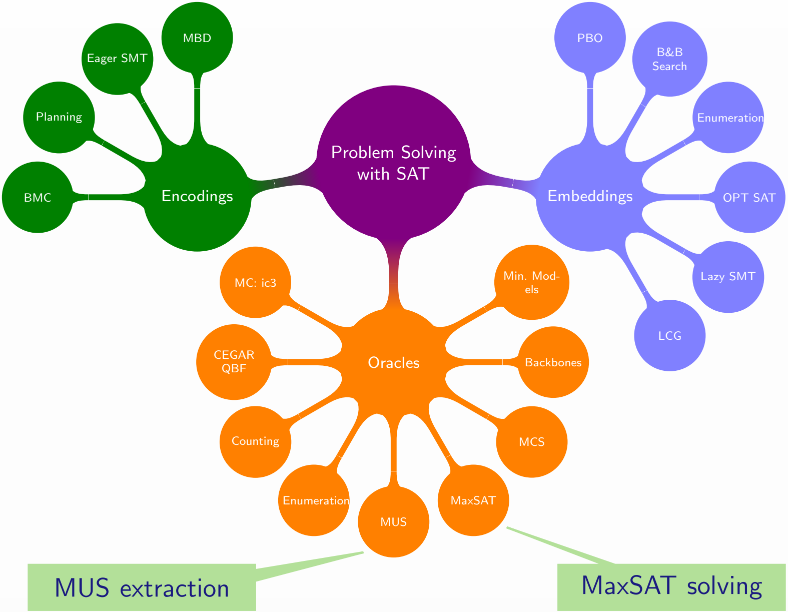


Selection of topics



Outline

Minimal Unsatisfiability

Maximum Satisfiability

Additional Exercises

Analyzing inconsistency – timetabling

Subject	Day	Time	Room
Intro Prog	Mon	9:00-10:00	6.2.46
Intro AI	Tue	10:00-11:00	8.2.37
Databases	Tue	11:00-12:00	8.2.37
... (hundreds of consistent constraints)			
Linear Alg	Mon	9:00-10:00	6.2.46
Calculus	Tue	10:00-11:00	8.2.37
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- Set of constraints consistent / satisfiable?

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- How to compute these **minimal** sets?

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ ($\models \bot$), $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \models \bot$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \bot$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- Given $\mathcal{F}$ $(\vDash \perp)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \perp$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \vDash \perp$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ $(\vDash \perp)$, $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \vDash \perp$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \perp$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- Given $\mathcal{F}$ $(\vDash \perp)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \perp$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \vDash \perp$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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- Given $\mathcal{F}$ ($\models \perp$), $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \models \perp$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \perp$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- MUSes and MCSes are (subset-)**minimal** sets
- MUSes and minimal hitting sets of MCSes and vice-versa [R87,...]

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ ($\vDash \bot$), $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \vDash \bot$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \bot$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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- MUSes and MCSes are (subset-)**minimal** sets
- MUSes and minimal hitting sets of MCSes and vice-versa [R87,...]
- How to compute MUSes & MCSes **efficiently** with SAT oracles?

Why it matters?

- Analysis of over-constrained systems
  - Model-based diagnosis [R87,...]
    - Software fault localization
    - Spreadsheet debugging
    - Debugging relational specifications (e.g. Alloy)
    - Type error debugging
    - Axiom pinpointing in description logics
    - ...
  - Model checking of software & hardware systems
  - Inconsistency measurement
  - Minimal models; MinCost SAT; ...
  - ...
- Find minimal relaxations to recover consistency
  - But also minimum relaxations to recover consistency, eg. **MaxSAT**
- **Find minimal explanations of inconsistency**
  - But also minimum explanations of inconsistency, eg. Smallest MUS

Deletion-based algorithm

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
    $\mathcal{M} \leftarrow \mathcal{F}$
    foreach $c \in \mathcal{M}$ **do**
        if $\neg\text{SAT}(\mathcal{M} \setminus \{c\})$ **then**
            $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\text{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
    return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$ [CD91,BDTW93]

Deletion-based algorithm

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
    $\mathcal{M} \leftarrow \mathcal{F}$
    foreach $c \in \mathcal{M}$ **do**
        if $\neg\text{SAT}(\mathcal{M} \setminus \{c\})$ **then**
            $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // Remove c from $\mathcal{M}$
    return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$ [CD91,BDTW93]

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$	Outcome

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\text{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\text{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \lor \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \lor \neg x_4)$	(x_3)	(x_4)	$(x_5 \lor x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6
$c_4..c_7$	$c_4..c_6$	1	Drop c_7

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6
$c_4..c_7$	$c_4..c_6$	1	Drop c_7

- MUS: $\{c_4, c_5, c_6\}$

Many MUS algorithms

- Formula $\mathcal{F}$ with m clauses k the size of largest minimal subset

Algorithm	Oracle Calls	Reference
Insertion-based	$\mathcal{O}(k\ m)$	[PS88,vMW08]
MCS_MUS	$\mathcal{O}(k\ m)$	[BK15]
Deletion-based	$\mathcal{O}(m)$	[CD91,BDTW93]
Linear insertion	$\mathcal{O}(m)$	[MSL'11,BLMS'12]
Dichotomic	$\mathcal{O}(k\ \log(m))$	[HLSB06]
QuickXplain	$\mathcal{O}(k + k\ \log(\frac{m}{k}))$	[J01,J04]
Progression	$\mathcal{O}(k\ \log(1 + \frac{m}{k}))$	[MSJB13,L14]

- **Note:** Lower bound in $\text{FP}_{||}^{\text{NP}}$ and upper bound in FP^{NP} [CT95]
- Oracle calls correspond to testing unsatisfiability with SAT solver
- Practical optimizations: clause set trimming; clause set refinement; redundancy removal; (recursive) model rotation

Outline

Minimal Unsatisfiability

Maximum Satisfiability

Additional Exercises

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest MCSes

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest MCSes
  - **Note**: Clauses can have weights & there can be hard clauses

Recap MaxSAT

$x_6 \lor x_2$	$\neg x_6 \lor x_2$	$\neg x_2 \lor x_1$	$\neg x_1$
$\neg x_6 \lor x_8$	$x_6 \lor \neg x_8$	$x_2 \lor x_4$	$\neg x_4 \lor x_5$
$x_7 \lor x_5$	$\neg x_7 \lor x_5$	$\neg x_5 \lor x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest **cost** MCSes
  - **Note**: Clauses can have weights & there can be hard clauses

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest **cost** MCSes
  - **Note**: Clauses can have weights & there can be hard clauses
- **Many** practical applications

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No		
	Yes		

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

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		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

- Must satisfy hard clauses, if any
- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost (s.t. hard & remaining soft clauses are satisfied)

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		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

- Must satisfy hard clauses, if any
- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost (s.t. hard & remaining soft clauses are satisfied)
- **Note**: goal is to compute **set** of satisfied (or falsified) clauses; **not** just the cost **!**

Issues with MaxSAT

- **Unit propagation is unsound for MaxSAT**

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  - Formula with all clauses soft:

$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

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  - After unit propagation:

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  - Is 2 the MaxSAT solution??

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  - **No!** Enough to either falsify (x) or (z)

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$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation

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  - **No!** Enough to either falsify (x) or (z)
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- Cannot learn clauses (using unit propagation)

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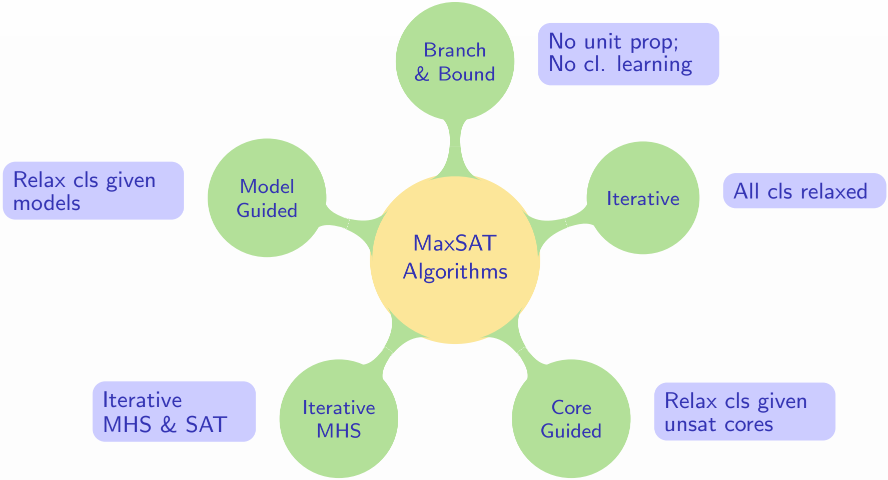
  - After unit propagation:

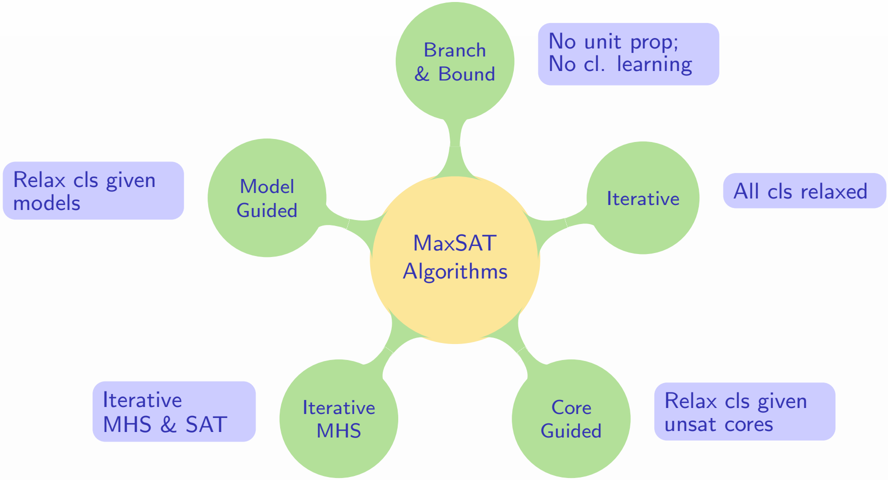
$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation
- Cannot learn clauses (using unit propagation)
- Need to solve MaxSAT using different techniques

Many MaxSAT approaches

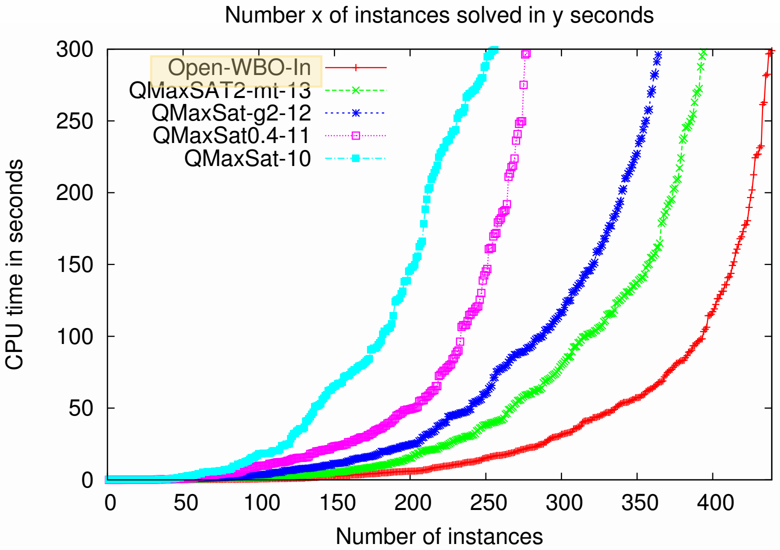


Many MaxSAT approaches



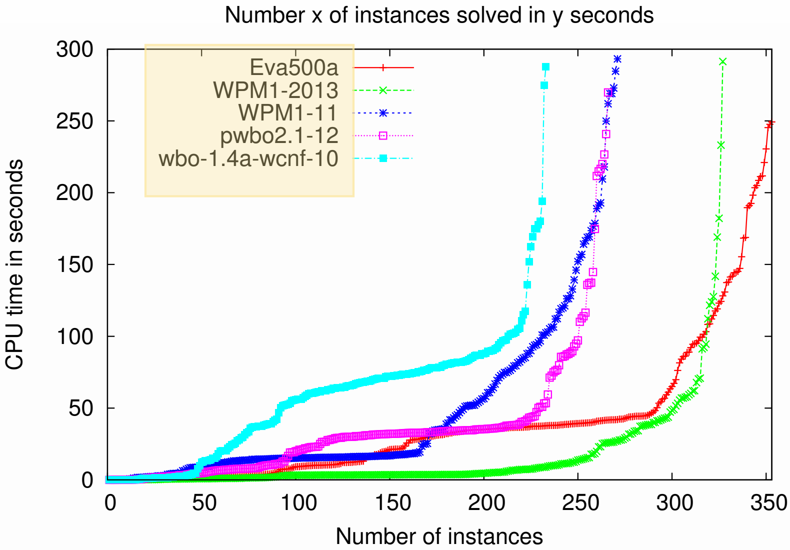
- For practical (**industrial**) instances: **core-guided** approaches are the most effective [MaxSAT14]

Core-guided solver performance – partial



Source: [MaxSAT 2014 organizers]

Core-guided solver performance – weighted partial



Source: [MaxSAT 2014 organizers]

Outline

Minimal Unsatisfiability

Maximum Satisfiability

- Iterative SAT Solving
- Core-Guided Algorithms
- Minimum Hitting Sets

Additional Exercises

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Example CNF formula

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\neg x_6 \vee x_2 \vee r_2$ $\neg x_2 \vee x_1 \vee r_3$ $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $x_6 \vee \neg x_8 \vee r_6$ $x_2 \vee x_4 \vee r_7$ $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\neg x_7 \vee x_5 \vee r_{10}$ $\neg x_5 \vee x_3 \vee r_{11}$ $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Relax all clauses; Set $UB = 12 + 1$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Formula is SAT; E.g. all $x_i = 0$ and $r_1 = r_7 = r_9 = 1$ (i.e. cost $= 3$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 2$

Refine $UB = 3$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 2$

Formula is SAT; E.g. $x_1 = x_2 = 1$; $x_3 = ... = x_8 = 0$ and $r_4 = r_9 = 1$ (i.e. cost $= 2$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad \neg x_6 \vee x_2 \vee r_2$ $\quad \neg x_2 \vee x_1 \vee r_3$ $\quad \neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad x_6 \vee \neg x_8 \vee r_6$ $\quad x_2 \vee x_4 \vee r_7$ $\quad \neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_{11}$ $\quad \neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Refine $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Formula is UNSAT; terminate

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad$ $\neg x_6 \vee x_2 \vee r_2$ $\quad$ $\neg x_2 \vee x_1 \vee r_3$ $\quad$ $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad$ $x_6 \vee \neg x_8 \vee r_6$ $\quad$ $x_2 \vee x_4 \vee r_7$ $\quad$ $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad$ $\neg x_7 \vee x_5 \vee r_{10}$ $\quad$ $\neg x_5 \vee x_3 \vee r_{11}$ $\quad$ $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

AtMost$k$/PB constraints over **all** relaxation variables

All (possibly many) soft clauses relaxed

Outline

Minimal Unsatisfiability

Maximum Satisfiability
- Iterative SAT Solving
- Core-Guided Algorithms
- Minimum Hitting Sets

Additional Exercises

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Example CNF formula

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Formula is UNSAT; OPT $\leq |\varphi| - 1$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\quad \neg x_6 \vee x_2$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\quad \neg x_7 \vee x_5$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost$k$, $k = 1$, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\quad$ $\neg x_6 \vee x_2$ $\quad$ $\neg x_2 \vee x_1 \vee r_1$ $\quad$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad$ $x_6 \vee \neg x_8$ $\quad$ $x_2 \vee x_4 \vee r_3$ $\quad$ $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\quad$ $\neg x_7 \vee x_5$ $\quad$ $\neg x_5 \vee x_3 \vee r_5$ $\quad$ $\neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Formula is (again) UNSAT; $\text{OPT} \leq |\varphi| - 2$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\neg x_6 \vee x_2 \vee r_8$ $\neg x_2 \vee x_1 \vee r_1$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $x_6 \vee \neg x_8$ $x_2 \vee x_4 \vee r_3$ $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\neg x_7 \vee x_5 \vee r_{10}$ $\neg x_5 \vee x_3 \vee r_5$ $\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Add new relaxation variables and update AtMost$k$, k=2, constraint

MSU3 core-guided algorithm

$x_6 \lor x_2 \lor r_7$ $\quad$ $\neg x_6 \lor x_2 \lor r_8$ $\quad$ $\neg x_2 \lor x_1 \lor r_1$ $\quad$ $\neg x_1 \lor r_2$

$\neg x_6 \lor x_8$ $\quad$ $x_6 \lor \neg x_8$ $\quad$ $x_2 \lor x_4 \lor r_3$ $\quad$ $\neg x_4 \lor x_5 \lor r_4$

$x_7 \lor x_5 \lor r_9$ $\quad$ $\neg x_7 \lor x_5 \lor r_{10}$ $\quad$ $\neg x_5 \lor x_3 \lor r_5$ $\quad$ $\neg x_3 \lor r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Instance is now SAT

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\neg x_6 \vee x_2 \vee r_8$ $\neg x_2 \vee x_1 \vee r_1$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $x_6 \vee \neg x_8$ $x_2 \vee x_4 \vee r_3$ $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\neg x_7 \vee x_5 \vee r_{10}$ $\neg x_5 \vee x_3 \vee r_5$ $\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Relaxed soft clauses become **hard**

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Some clauses not relaxed

Relaxed soft clauses become **hard**

Outline

Minimal Unsatisfiability

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- Iterative SAT Solving
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Additional Exercises

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad$ $c_2 = \neg x_6 \vee x_2$ $\quad$ $c_3 = \neg x_2 \vee x_1$ $\quad$ $c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad$ $c_6 = x_6 \vee \neg x_8$ $\quad$ $c_7 = x_2 \vee x_4$ $\quad$ $c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad$ $c_{10} = \neg x_7 \vee x_5$ $\quad$ $c_{11} = \neg x_5 \vee x_3$ $\quad$ $c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)?

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$c_1 = x_6 \lor x_2 \qquad c_2 = \neg x_6 \lor x_2 \qquad c_3 = \neg x_2 \lor x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \lor x_8 \qquad c_6 = x_6 \lor \neg x_8 \qquad c_7 = x_2 \lor x_4 \qquad c_8 = \neg x_4 \lor x_5$$

$$c_9 = x_7 \lor x_5 \qquad c_{10} = \neg x_7 \lor x_5 \qquad c_{11} = \neg x_5 \lor x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad$ $c_2 = \neg x_6 \vee x_2$ $\quad$ $c_3 = \neg x_2 \vee x_1$ $\quad$ $c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad$ $c_6 = x_6 \vee \neg x_8$ $\quad$ $c_7 = x_2 \vee x_4$ $\quad$ $c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad$ $c_{10} = \neg x_7 \vee x_5$ $\quad$ $c_{11} = \neg x_5 \vee x_3$ $\quad$ $c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
- Core of $\mathcal{F}$: $\{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
- Core of $\mathcal{F}$: $\{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

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MHS approach for MaxSAT

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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)?

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)? Yes
- Terminate & return 2

MaxSAT solving with SAT oracles – a sample

- A sample of recent algorithms:

Algorithm	# Oracle Queries	Reference
Linear search SU	Exponential***	[e.g. LBP10]
Binary search	Linear*	[e.g. FM06]
FM/WMSU1/WPM1	Exponential**	[FM06,MSM08,MMSP09,ABL09a,ABGL12]
WPM2	Exponential**	[ABL10,ABGL13]
Bin-Core-Dis	Linear	[HMMS11,MHMS12]
Iterative MHS	Exponential	[DB11,DB13a,DB13b]

  * $\mathcal{O}(\log m)$ queries with SAT oracle, for (partial) unweighted MaxSAT

 ** Weighted case; depends on computed cores

*** On # bits of problem instance (due to weights)

- But also additional recent work:
  - Progression
  - Soft cardinality constraints (OLL)
  - MaxSAT resolution
  - ...

Outline

Minimal Unsatisfiability

Maximum Satisfiability

Additional Exercises

Exercise – How many MCSes & MUSes can there be?

- Give example showing that lower bound on largest number of MCSes is exponential on formula size
  - Hint: Simply suggest formula with exponentially large number of MaxSAT solutions

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- Give example showing that lower bound on largest number of MUSes is exponential on formula size

Solution – number of MCSes

$$\begin{array}{cc} (x_1) & (\neg x_1) \\ (x_2) & (\neg x_2) \\ \dots & \dots \\ (x_n) & (\neg x_n) \end{array}$$

Solution – number of MCSes

$$\begin{array}{cc} (x_1) & (\neg x_1) \\ (x_2) & (\neg x_2) \\ \dots & \dots \\ (x_n) & (\neg x_n) \end{array}$$

- For each $i = 1, \dots, n$ either pick (x_i) or $(\neg x_i)$, i.e. 2 cases

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- For each $i = 1, \dots, n$ either pick (x_i) or $(\neg x_i)$, i.e. 2 cases
- Thus, 2^n MCSes

Solutions – number of MUSes I

$$(\neg x_1) \wedge (x_1 \vee z_1)$$
$$(\neg y_1) \wedge (y_1 \vee z_1)$$
$$\ldots$$
$$(\neg x_n) \wedge (x_n \vee z_n)$$
$$(\neg y_n) \wedge (y_n \vee z_n)$$

$$(\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n)$$

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$$(\neg y_1) \wedge (y_1 \vee z_1)$$
$$\ldots \qquad (\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n)$$
$$(\neg x_n) \wedge (x_n \vee z_n)$$
$$(\neg y_n) \wedge (y_n \vee z_n)$$

- For each $i = 1, \ldots, n$ either resolve away x_i or y_i, i.e. 2 cases

Solutions – number of MUSes I

$$
\begin{array}{ll}
(\neg x_1) \wedge (x_1 \vee z_1) & \\
(\neg y_1) \wedge (y_1 \vee z_1) & \\
\ldots & (\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n) \\
(\neg x_n) \wedge (x_n \vee z_n) & \\
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\end{array}
$$

- For each $i = 1, \ldots, n$ either resolve away x_i or y_i, i.e. 2 cases
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\end{array}
$$

- For each $i = 1, \ldots, n$ either resolve away x_i or y_i, i.e. 2 cases
- Thus, 2^n MUSes
- But, there exist formulas with more MUSes. **How?**

Solutions – number of MUSes II

$$(\neg x_1) \wedge (\neg x_2) \wedge \ldots \wedge (\neg x_r)$$
$$(x_1 \vee z_1) \wedge (x_2 \vee z_1) \wedge \ldots \wedge (x_r \vee z_1)$$
$$(x_1 \vee z_2) \wedge (x_2 \vee z_2) \wedge \ldots \wedge (x_r \vee z_2)$$
$$\ldots$$
$$(x_1 \vee z_n) \wedge (x_2 \vee z_n) \wedge \ldots \wedge (x_r \vee z_n)$$
$$(\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n)$$

Solutions – number of MUSes II

$$
\begin{array}{l}
(\neg x_1) \wedge (\neg x_2) \wedge \ldots \wedge (\neg x_r) \\
(x_1 \vee z_1) \wedge (x_2 \vee z_1) \wedge \ldots \wedge (x_r \vee z_1) \\
(x_1 \vee z_2) \wedge (x_2 \vee z_2) \wedge \ldots \wedge (x_r \vee z_2) \\
\ldots \\
(x_1 \vee z_n) \wedge (x_2 \vee z_n) \wedge \ldots \wedge (x_r \vee z_n) \\
(\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n)
\end{array}
$$

- There are r^n MUSes

Solutions – number of MUSes II

$$
\begin{aligned}
&(\neg x_1) \wedge (\neg x_2) \wedge \ldots \wedge (\neg x_r)\\
&(x_1 \vee z_1) \wedge (x_2 \vee z_1) \wedge \ldots \wedge (x_r \vee z_1)\\
&(x_1 \vee z_2) \wedge (x_2 \vee z_2) \wedge \ldots \wedge (x_r \vee z_2)\\
&\ldots\\
&(x_1 \vee z_n) \wedge (x_2 \vee z_n) \wedge \ldots \wedge (x_r \vee z_n)\\
&(\neg z_1 \vee \neg z_2 \vee \ldots \vee \neg z_n)
\end{aligned}
$$

- There are r^n MUSes
- Upper bound by Sperner's theorem: $C(m, \lfloor \frac{m}{2} \rfloor)$

Exercise – Sudoku puzzle generator

- Randomly generate valid Sudoku puzzles starting from a filled grid

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  - Iteratively (and randomly) punch holes in the Sudoku grid
    - If Sudoku puzzle is still valid, i.e. number of solutions is 1, then repeat loop
    - Otherwise, output Sudoku grid with most recently punched hole removed

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  - How many SAT oracles calls?

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  - How many SAT oracles calls?
    - Linear number of calls on number of cells in Sudoku puzzle

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    - Otherwise, output Sudoku grid with most recently punched hole removed
  - How many SAT oracles calls?
    - Linear number of calls on number of cells in Sudoku puzzle
  - Can we do better?

Some final notes

- SAT is a low-level, but very powerful problem solving paradigm
- There is an ongoing revolution on problem solving with SAT oracles
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.

Some final notes

- SAT is a low-level, but very powerful problem solving paradigm
- There is an ongoing revolution on problem solving with SAT oracles
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.
- Many fascinating research topics out there !

Links for tools

- SAT solvers:
  - minisat: https://github.com/niklasso/minisat
  - glucose: http://www.labri.fr/perso/lsimon/glucose/
- MaxSAT solvers:
  - MSCG: http://logos.ucd.ie/web/doku.php?id=mscg
  - OpenWBO: http://sat.inesc-id.pt/open-wbo/
  - MaxHS: http://www.maxhs.org
- MCS extractors:
  - mcsXL: http://logos.ucd.ie/wiki/doku.php?id=mcsxl
  - LBX: http://logos.ucd.ie/wiki/doku.php?id=lbx
  - MCSls: http://logos.ucd.ie/wiki/doku.php?id=mcsls
- MUS extractors:
  - MUSer: http://logos.ucd.ie/wiki/doku.php?id=muser
- Many other tools available from:
  http://logos.ucd.ie/wiki/doku.php?id=soft

Thank You