Symmetries of maximal supergravity in the light-cone gauge

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(with Sudarshan Ananth and Lars Brink)

Why $(\mathcal{N}=8, \mathrm{d}=4)$ supergravity?

- Maximum supersymmetry allowed for spin 2 : $\mathcal{N}=8$ (32 supercharges)

$$128_b + 128_f = 256 \text{ physical d.o.f.}$$

Helicity	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
Field	h	$\overline{\psi}_m$	$\overline{A}_{mn}$	$\overline{\chi}_{mnp}$	C_{mnpq}	χ^{mnp}	A^{mn}	ψ^m	$\overline{h}$
	1	8	28	56	70	56	28	8	1

$m, n.. = 1, 2, \ldots, 8$ are SU(8) indices

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- Lot of symmetries : Maximal SUSY, SU(8) R-symmetry, $E_{7(7)}$ symmetry

[E. Cremmer and B. Julia, 1979]

Symmetry $\rightarrow$ Conservation law $\rightarrow$ Constraints

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Symmetry $\rightarrow$ Conservation law $\rightarrow$ Constraints

- Better UV behaviour than expected; **finite up to four loops**

KLT relations : $\mathcal{N} = 8$ supergravity $\sim$ ($\mathcal{N} = 4$ superYang-Mills theory)^2

[K. Stelle, 2007 ; Z. Bern et. al., 2009]

More symmetry than meets the eye??

Exceptional symmetries in maximal supergravity

$$(\mathcal{N}=1, \mathrm{d}=11) \text{ supergravity}$$

$$\Big\downarrow$$

$$(\mathcal{N}=8, \mathrm{d}=5) \text{ supergravity} \quad \rightarrow \quad E_{6(6)}$$

$$(\mathcal{N}=8, \mathrm{d}=4) \text{ supergravity} \quad \rightarrow \quad E_{7(7)} \quad : \ 133$$

$$(\mathcal{N}=16, \mathrm{d}=3) \text{ supergravity} \quad \rightarrow \quad E_{8(8)} \quad : \ 248$$

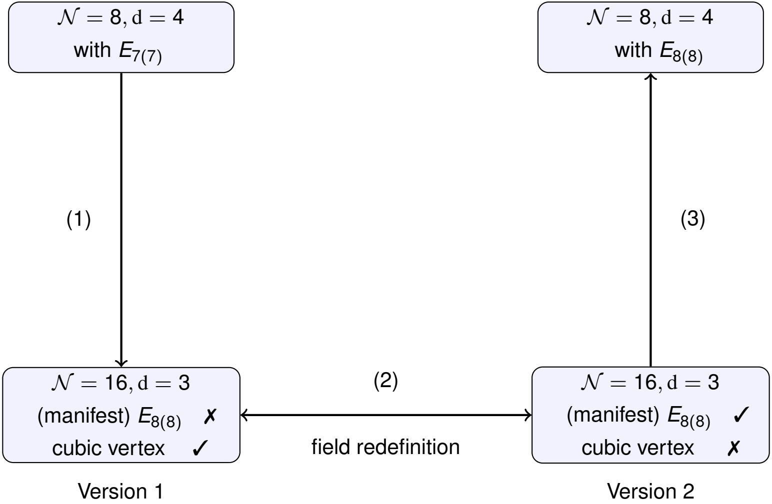
$$\Big\downarrow$$

$$(\mathcal{N}=32, \mathrm{d}=1) \text{ supergravity} \quad \rightarrow \quad E_{10}/E_{11}{}^{\dagger}$$

† [P. C. West, 2001; T. Damour, M. Henneaux and H. Nicolai, 2002]

$$\boxed{\text{Can the } E_{7(7)} \text{ be enhanced to a bigger symmetry?}}$$

Method



$\mathcal{N}=8, \mathrm{d}=4$
with $E_{7(7)}$
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(1)
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(2)
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cubic vertex ✓
field redefinition
$\mathcal{N}=16, \mathrm{d}=3$
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Version 1
Version 2

Notation

- We use natural units and the metric signature $(-,+,+,+)$.

- Light-cone coordinates :

$$x^{\pm} \; = \; \frac{x^0 \; \pm \; x^3}{\sqrt{2}} \; , \quad x \; = \; \frac{x^1 \; + \; i\, x^2}{\sqrt{2}} \; ; \quad \bar{x} \; = \; \frac{x^1 \; + \; i\, x^2}{\sqrt{2}}$$

Derivatives : $\partial_{\pm}\;, \partial\,, \bar{\partial}$

$$x^{+} \; : \; \text{light-cone time} \quad \Rightarrow \quad i\,\frac{\partial}{\partial x^{+}} = P_{+} \; : \; \text{Hamiltonian}$$

Why the light-cone gauge?

- Deals with the physical degrees of freedom only.
- On-shell scattering amplitude structures appear at the level of Lagrangian.

Light-cone superspace

- Add Grassmann variables to the spacetime

$$(x^+ , x^- , x , \bar{x} , \theta^m , \bar{\theta}_n) , \quad m, n = 1, 2, ..., 8$$

- Light-cone SUSY generators (built out of $\theta^m, \bar{\theta}_m$)

Kinematical : $q_+^m , \quad \bar{q}_{+n}$

$$\{ q_+^m , \bar{q}_{+n} \} = \sqrt{2}\, \delta^m{}_n \, P_-$$

Dynamical : $q_-^m , \quad \bar{q}_{-m}$

$$\{ q_-^m , \bar{q}_{-n} \} = \sqrt{2}\, \delta^m{}_n \, P_+$$

Can be used to construct the Hamiltonian!

- Chiral derivatives :

$$d^m = - \frac{\partial}{\partial \bar{\theta}_m} - i \sqrt{2}\, \theta^m \, \partial^+ , \quad \bar{d}_n = (d^n)^*$$

[A.K.H. Bengtsson, I. Bengtsson and L. Brink, 1983]

N=8 supergravity in light-cone gauge

- Light-cone superfield

$$\phi \;=\; \frac{1}{\partial^{+2}}\,h \,+\, i\,\theta^{m}\,\frac{1}{\partial^{+2}}\,\overline{\psi}_m \,+\, i\,\theta^{mn}\,\frac{1}{\partial^{+}}\,\overline{A}_{mn} \,-\, \theta^{mnp}\,\frac{1}{\partial^{+}}\,\overline{\chi}_{mnp} \,-\, \theta^{mnpq}\,\overline{C}_{mnpq}$$

$$+\,\epsilon_{mnpqrstu}\big[\,i\,\theta^{mnpqr}\;\chi^{stu} \,+\, i\,\theta^{mnpqrs}\,\partial^{+}A^{tu} \,+\, \theta^{mnpqrst}\,\partial^{+}\psi^{u} \,+\, 4\,\theta^{mnpqrstu}\,\partial^{+2}\overline{h}\,\big]$$

where

$$\theta^{m_1\ldots m_n} \;=\; \frac{1}{n!}\,\theta^{m_1}\ldots\theta^{m_n}$$

- The action to order κ

$$\mathcal{S} = -\frac{1}{64}\int d^4x\,d^8\theta\,d^8\overline{\theta}\,\left\{-\overline{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \,-\, 2\,\kappa\,\left(\frac{1}{\partial^{+2}}\,\overline{\phi}\,\overline{\partial}\phi\,\overline{\partial}\phi \,+\, \frac{1}{\partial^{+2}}\,\phi\,\partial\overline{\phi}\,\partial\overline{\phi}\right)\right\}$$

$$\Box \;=\; 2\,(\,\partial\,\overline{\partial} \,-\, \partial^{+}\partial^{-})$$

[S. Ananth, L. Brink and P. Ramond, 2005]

$E_{7(7)}$ symmetry

- Covariantly, $E_{7(7)}$ acts on scalars and vector fields only. It is a symmetry of the equations of motion.

$$
\begin{array}{ccccc}
E_{7(7)} & = & SU(8) & \times & \dfrac{E_{7(7)}}{SU(8)} \\
133 & & 63 \ \text{R-symmetry} & & 70 \ \text{coset transformations}
\end{array}
$$

$$
\left[\delta_{SU(8)}\,,\delta_{SU(8)}\right] \;=\; \delta_{SU(8)}\;, \qquad \left[\,\delta_{\frac{E_{7(7)}}{SU(8)}}\,,\,\delta_{\frac{E_{7(7)}}{SU(8)}}\,\right]\phi \;=\; \delta_{SU(8)}\,\phi
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- In the light-cone formalism,

$$\delta_{SU(8)}\,\phi \;=\; -\frac{i}{\sqrt{2}\partial^+}\,\omega^n{}_m\left(q^m\,\bar{q}_n \,-\, \frac{1}{8}\,\delta^m{}_n\,q^p\,\bar{q}_p\right)\phi$$

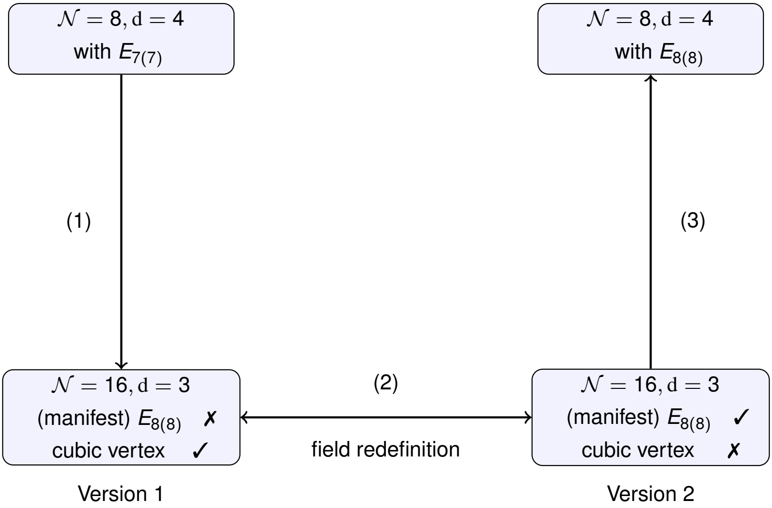
$$\delta_{\frac{E_{7(7)}}{SU(8)}}\,\phi \;=\; -\frac{2}{\kappa}\,\theta^{mnpq}\,\Xi_{mnpq} \,+\, \frac{\kappa}{4!}\,\Xi^{mnpq}\,\frac{1}{\partial^{+2}}\left(\overline{d}_m\overline{d}_n\overline{d}_p\overline{d}_q\,\frac{1}{\partial^+}\,\phi\,\partial^{+3}\phi \,+\, \ldots\right)$$

[L. Brink, S.S. Kim and P. Ramond, 2008]

$\rightarrow$ $E_{7(7)}$ acts on all the physical fields!

$\rightarrow$ It is a symmetry of the action $\rightarrow$ as fundamental as SUSY

Method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cubic vertex ✓
field redefinition
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Version 1
Version 2

Step 1 : From $\mathrm{d} = 4$ to $\mathrm{d} = 3$

Drop the $\bar{x}$ coordinate :

$$(x^+,\, x^-,\, x,\, \bar{x}\,) \quad \longrightarrow \quad (x^+,\, x^-,\, x)$$

Field content : 128 scalars + 128 spin-$\frac{1}{2}$ fermions $\longrightarrow$ same ϕ can be used

$\mathcal{N} = 8, \mathrm{d} = 4$ supergravity

$$\mathcal{L}_{d=4} \;=\; -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; 2\,\kappa\left(\frac{1}{\partial^{+2}}\,\bar{\phi}\;\bar{\partial}\phi\;\bar{\partial}\phi \;+\; \frac{1}{\partial^{+2}}\,\phi\;\partial\bar{\phi}\;\partial\bar{\phi}\right)$$

$$\Big\downarrow \text{ drop the } \bar{\partial}$$

$\mathcal{N} = 16, \mathrm{d} = 3$ supergravity

$$\mathcal{L}_{d=3} \;=\; -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; 2\,\kappa\,(\frac{1}{\partial^{+2}}\;\bar{\phi}\;\;\partial\,\phi\,\partial\,\phi + \frac{1}{\partial^{+2}}\;\phi\,\partial\,\bar{\phi}\,\partial\,\bar{\phi})$$

- with a **cubic vertex**
- without (manifest) $E_{8(8)}$ symmetry

Version 1

$\mathcal{N} = 16, \mathrm{d} = 3$ supergravity : Version 2

$$
\begin{array}{ccc}
(\mathcal{N} = 1, \mathrm{d} = 11) \ \text{supergravity} & & \\
\Big\downarrow & & \\
(\mathcal{N} = 8, \mathrm{d} = 5) \ \text{supergravity} & \rightarrow & E_{6(6)} \\
(\mathcal{N} = 8, \mathrm{d} = 4) \ \text{supergravity} & \rightarrow & E_{7(7)} \\
\boxed{(\mathcal{N} = 16, \mathrm{d} = 3) \ \text{supergravity} \quad \rightarrow \quad E_{8(8)}} & & \\
\Big\downarrow & & \\
(\mathcal{N} = 32, \mathrm{d} = 1) \ \text{supergravity} & \rightarrow & E_{10}/E_{11}{}^{\dagger}
\end{array}
$$

- Field content : ϕ = 128 scalars + 128 spin-$\frac{1}{2}$ fermions
- $SO(16)$ R-symmetry $\Rightarrow$ Lagrangian has **no cubic vertex** (or odd-order vertices).
- Manifest $E_{8(8)}$ symmetry

[N.Marcus and J.H.Schwarz, 1983]

$\mathcal{N} = 16, \mathrm{d} = 3$ supergravity : Version 2

- $E_{8(8)}$ symmetry

$$
\begin{array}{cccc}
E_{8(8)} & = \quad SO(16) & \times & \dfrac{E_{8(8)}}{SO(16)} \\
248 & 120 \text{ R-symmetry} & & 128 \text{ coset transformations}
\end{array}
$$

$$
\left[\delta_{SO(16)}\,,\ \delta_{SO(16)}\right]\,\phi \;=\; \delta_{SO(16)}\,\phi\,, \qquad \left[\,\delta_{\frac{E_{8(8)}}{SO(16)}}\,,\ \delta_{\frac{E_{8(8)}}{SO(16)}}\,\right]\phi = \delta_{SO(16)}\,\phi
$$

- In the light-cone superspace,

$$
SO(16) \quad : \quad T^{mn}\,\phi \;=\; \frac{q_+^m q_+^n}{\partial^+}\,\phi\,, \quad T\,\phi \;=\; \frac{i}{4\sqrt{2}\partial^+}\,[q_+^k, \bar{q}_{+k}]\,\phi \quad \text{etc.}
$$

$$
\text{coset} \quad : \quad \delta_{\frac{E_{8(8)}}{SO(16)}}\,\phi \;=\; \frac{1}{\kappa}\,F \;+\; \mathcal{O}(\kappa)
$$

$\rightarrow$ $E_{8(8)}$ can be represented on the same supermultiplet as $E_{7(7)}$ symmetry!

$\rightarrow$ $E_{8(8)}$ is a symmetry of the action.

[L. Brink, S.S. Kim and P. Ramond, 2008]

Step 2 : Version 1 $\longleftrightarrow$ Version 2

Version 2 :

$SO(16)$ R-symmetry $\Longrightarrow$ Lagrangian has no cubic vertex.

$$\mathcal{L} \quad = \quad -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;+\; \cancel{\mathcal{O}(\kappa)} \;+\; \mathcal{O}(\kappa^2)$$

Step 2 : Version 1 $\longleftrightarrow$ Version 2

Version 2 :

$SO(16)$ R-symmetry $\Longrightarrow$ Lagrangian has no cubic vertex.

$$\mathcal{L} \quad = \quad -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;+\; \cancel{\mathcal{O}(\kappa)} \;+\; \mathcal{O}(\kappa^2)$$

$$\downarrow$$

Field redefinition :

$$\phi \rightarrow \phi \quad = \quad \phi' \;+\; \frac{1}{3}\,\kappa\,(\partial^+\phi'\,\partial^+\phi') \;+\; \frac{2}{3}\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\,\phi'\,\partial^+\bar{\phi}'\right)$$

$$\downarrow$$

Version 1 :

$$\mathcal{L}' \quad = \quad -\bar{\phi}'\,\frac{\Box}{\partial^{+4}}\,\phi' \;-\; 2\,\kappa\,(\,\frac{1}{\partial^{+2}}\,\bar{\phi}'\;\;\bar{\partial}\,\phi'\,\bar{\partial}\,\phi' \;+\; \frac{1}{\partial^{+2}}\,\phi'\,\partial\,\bar{\phi}'\,\partial\,\bar{\phi}') \;+\; \mathcal{O}(\kappa^2)$$

This Lagrangian respects both the $SO(16)$ and $E_{8(8)}$ symmetry as in version 2.

[S. Ananth, L. Brink and S. Majumdar, arXiv:1711.09110]

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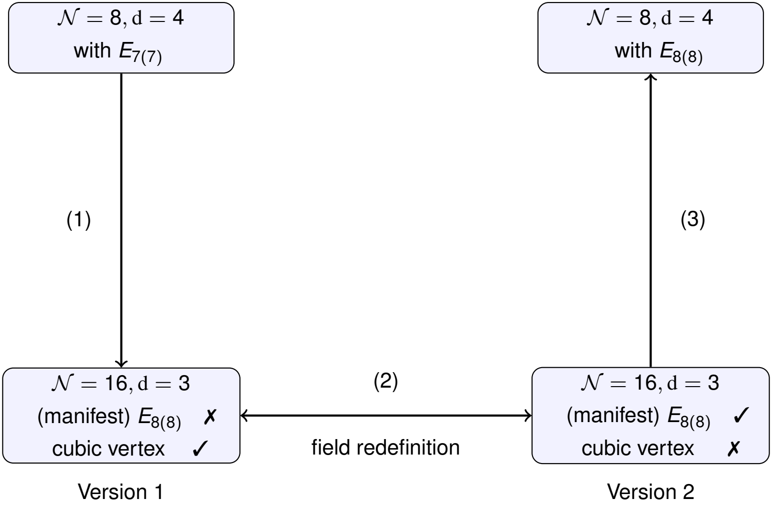
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Version 2

Step 3 : Back to $\mathrm{d} = 4$ with an $E_{8(8)}$ symmetry

$$(x^+ , x^- , x) \quad \rightarrow \quad (x^+ , x^- , x , \bar{x}) \ : \ \text{Put back } \bar{\partial}$$

- $\mathcal{N} = 16, \mathrm{d} = 3$ supergravity (Version 2) : Manifestly $E_{8(8)}$ invariant

$$\bar{q}_{-m}\phi \quad = \quad \frac{\partial}{\partial^+}\, \bar{q}_{+m}\, \phi \ + \ \mathcal{O}(\kappa^2)$$

[L. Brink, S.S. Kim and P. Ramond, 2008]

$$\Big\downarrow \quad \text{Replace } \partial \ \rightarrow \ \nabla\ (\partial\, , \bar{\partial})$$

- $\mathcal{N} = 8, \mathrm{d} = 4$ supergravity

$$\bar{q}_{-m}\phi \quad = \quad \frac{\nabla}{\partial^+}\, \bar{q}_{+m}\, \phi \ + \ \mathcal{O}(\kappa^2)$$

Recall : $\{ \, q^m_- \, , \, \bar{q}_{-n} \, \} \ = \ \sqrt{2}\, \delta^m{}_n \, P_+ \ \rightarrow$ Hamiltonian

$$\boxed{\text{A } SO(16) \text{ and } E_{8(8)} \text{ invariant theory of maximal supergravity in } \mathrm{d} = 4!}$$

[S. Ananth, L. Brink and S. Majumdar, arXiv:1711.09110]

Take-home message(s)

There could be more symmetries lurking in the $\mathcal{N} = 8$ thoery.

In the light-cone formalism, the exceptional symmetries are as important as supersymmetry. Which one is more fundamental?

Appendix

- Kinematical supersymmetries

$$q_+^m \;=\; -\frac{\partial}{\partial\,\bar\theta^m} \;+\; \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+\,; \quad \bar q_{+n} \;=\; \frac{\partial}{\partial\,\theta^n} \;-\; \frac{i}{\sqrt{2}}\bar\theta_n\,\partial^+$$

- Dynamical supersymmetries

$$q_-^m \;=\; \frac{\bar\partial}{\partial^+}\,q_+^m\,, \quad \bar q_{-n} \;=\; \frac{\partial}{\partial^+}\,\bar q_{+m}$$

- The chirality condition

$$d^m\,\phi\,(y) \;=\; 0\,, \quad \bar d_n\,\bar\phi\,(y) \;=\; 0\,,$$

- The inside-out constraint

$$\phi \;=\; \frac{1}{4}\,\frac{(d)^8}{\partial^{+4}}\,\bar\phi\,,$$

  where $(d)^8 \;=\; d^1\,d^2\,...\,d^8$

- Corrections to dynamical SUSY

$$\bar q_{-m}\,\phi \;=\; \kappa\,\frac{1}{\partial^+}(\bar\partial\,\bar q_{+m}\,\phi\,\partial^{+2}\,\phi \;-\; \partial^+\,\bar q_{+m}\,\phi\,\partial^+\,\bar\partial\,\phi)\,.$$

$E_{8(8)}$ in terms of $E_{7(7)}$

$$E_{8(8)} = SO(16) \times \frac{E_{8(8)}}{SO(16)}$$

248   120 R-symmetry   128 coset transformations

θ^m and $\bar{\theta}_m$, form a **16** representation

$$SO(16) \supset SU(8) \times U(1)\,, \qquad \mathbf{16} = \mathbf{8} + \bar{\mathbf{8}}\,.$$

The 120 $SO(16)$ transformations are decomposed in terms of $SU(8) \times U(1)$

$$\mathbf{120} = \mathbf{63_0} + \mathbf{28_{-1}} + \overline{\mathbf{28}}\mathbf{_1} + \mathbf{1_0}$$

$$\mathbf{63_0} = \delta_{\mathbf{SU(8)}}$$

Non-linearly realized coset $E_{8(8)}/SO(16)$ in terms of $SU(8) \times U(1)$ representations

$$\mathbf{128} = \mathbf{1'_2} + \mathbf{28'_1} + \mathbf{70_0} + \overline{\mathbf{28}}'_{\mathbf{-1}} + \bar{\mathbf{1}}'_{\mathbf{-2}} \quad .$$

$$\mathbf{70_0} = \delta_{\frac{\mathbf{E_{7(7)}}}{\mathbf{SU(8)}}}$$

128 $E_{8(8)}/SO(16)$ coset transformations

$$\delta_{\frac{E_{[}8(8)}{SO(16)}}\,\phi \quad = \quad \frac{1}{\kappa}\,F\,+\,\kappa\,\epsilon^{i_1\,i_2\cdots i_8}\,\sum_{c=-2}^{2}\left(\hat{\bar{d}}_{i_1\,i_2\cdots i_{2(c+2)}}\,\partial^{+c}\,F\right)$$
$$\times\left\{\left(\frac{\partial}{\partial\,\eta}\right)_{i_{2c+5}\cdots i_8}\,\partial^{+(c-2)}\left(e^{\eta\cdot\hat{\bar{d}}}\,\partial^{+(3-c)}\phi\,e^{-\eta\cdot\hat{\bar{d}}}\partial^{+(3-c)}\phi\right)\Big|_{\eta=0}\,+\,\mathcal{O}(\kappa^2)\right\},$$

where the sum is over the $U(1)$ charges $c = 2, 1, 0 - 1, -2$ of the bosonic fields, and

$$F \quad = \quad \frac{1}{\partial^{+2}}\,\beta\,(y^-)\;+\;i\,\theta^{mn}\,\frac{1}{\partial^+}\,\overline{\beta}_{mn}\,(y^-)\,-\,\theta^{mnpq}\,\overline{\beta}_{mnpq}\,(y^-)\,+$$
$$+\,i\widetilde{\theta}_{\,mn}\,\partial^+\,\beta^{mn}\,(y^-)\,+\,4\,\widetilde{\theta}\,\partial^{+2}\,\bar{\beta}\,(y^-)\;,$$

and

$$\hat{\bar{d}}_{i_1\,i_2\cdots i_{2(c+2)}}\;\equiv\;\hat{\bar{d}}_{i_1}\,\hat{\bar{d}}_{i_2}\,\cdots\,\hat{\bar{d}}_{2(c+2)}\;.$$

$$\delta_s^{dyn\,(2)}\phi \quad = \quad \frac{\kappa^2}{2}\,\sum_{c=-2}^{2}\,\frac{1}{\partial^{+(c+4)}}\left\{\frac{\partial}{\partial a}\,\frac{\partial}{\partial b}\,\left(\frac{\partial}{\partial\eta}\right)_{i_1\,i_2\cdots i_{2(c+2)}}\left(E\partial^{+(c+5)}\phi\,E^{-1}\right)\Big|_{a=b=\eta=0}\right.$$
$$\left.\times\,\frac{\epsilon^{i_1\,i_2\cdots i_8}}{(4-2c)!}\,\left(\frac{\partial}{\partial\eta}\right)_{i_{2c+5}\cdots i_8}\,\partial^{+2c}\,\left(E\partial^{+(4-c)}\phi E^{-1}\,\partial^{+(4-c)}\phi\right)\Big|_{\eta=0}\right\}\,.$$