Shear and Bulk viscosity in semi-QGP region

for pure glue theory

Najmul Haque

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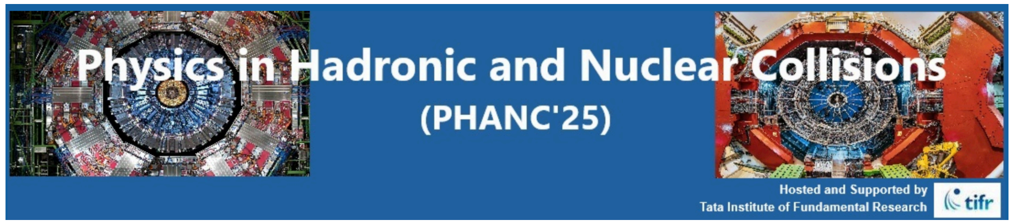


Shear and Bulk viscosity in semi-QGP region

for pure glue theory

Najmul, Rob, Manas, Ritesh

@NISER, Bhubaneswar (Feb 2023)

Yoshimasa

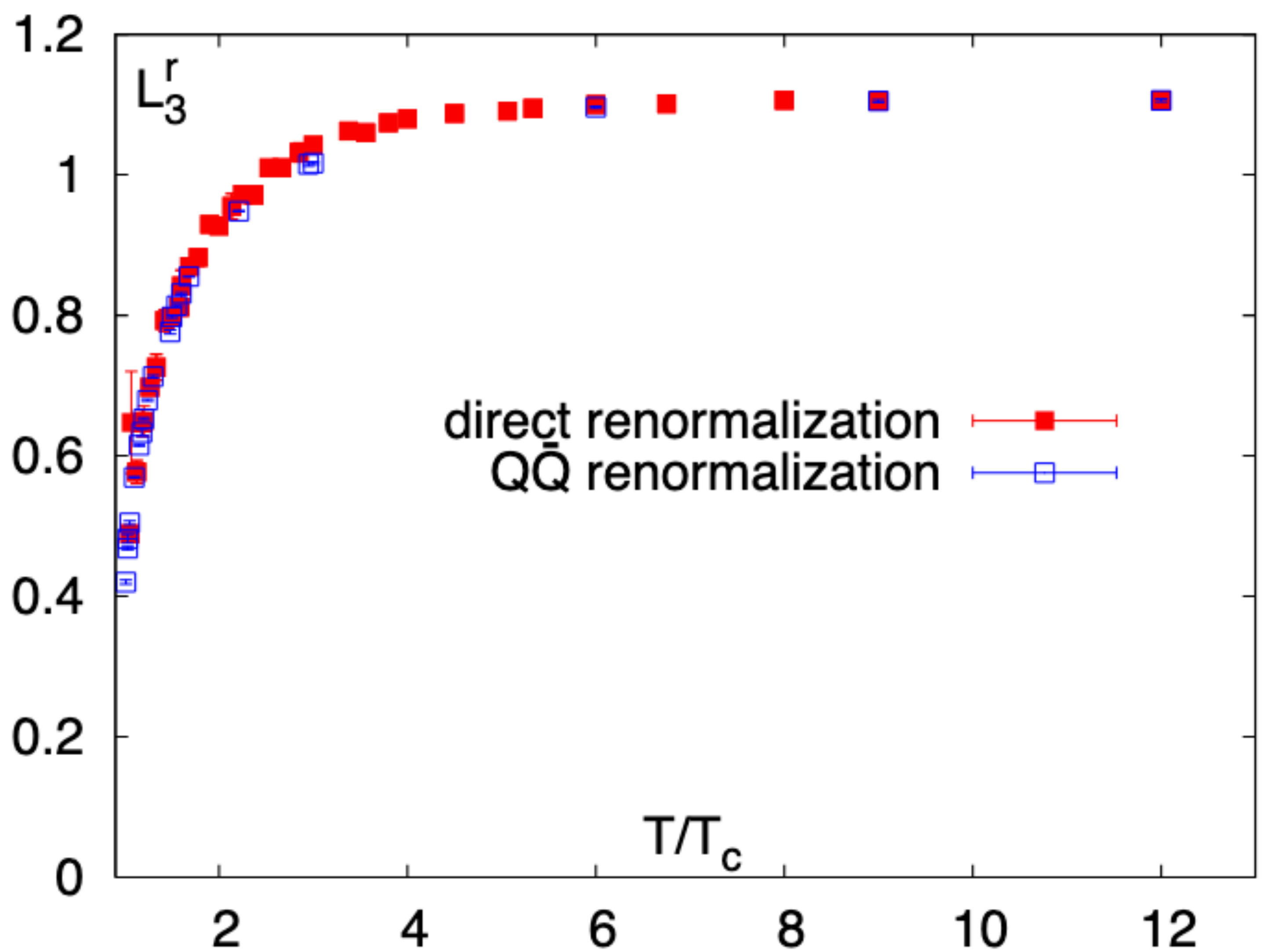
Polyakov loop

- In a $SU(N_c)$ gauge theory without dynamical quarks, the simplest way to measure the deconfining phase transition is through the Polyakov loop ℓ.

- The Polyakov loop is the trace of the Wilson line in the direction of imaginary time $\ell = \frac{1}{N_c} Tr \left[\mathscr{P} \exp \left(ig \int_0^{1/T} A_0(\mathbf{x}, \tau) d\tau \right) \right]$.

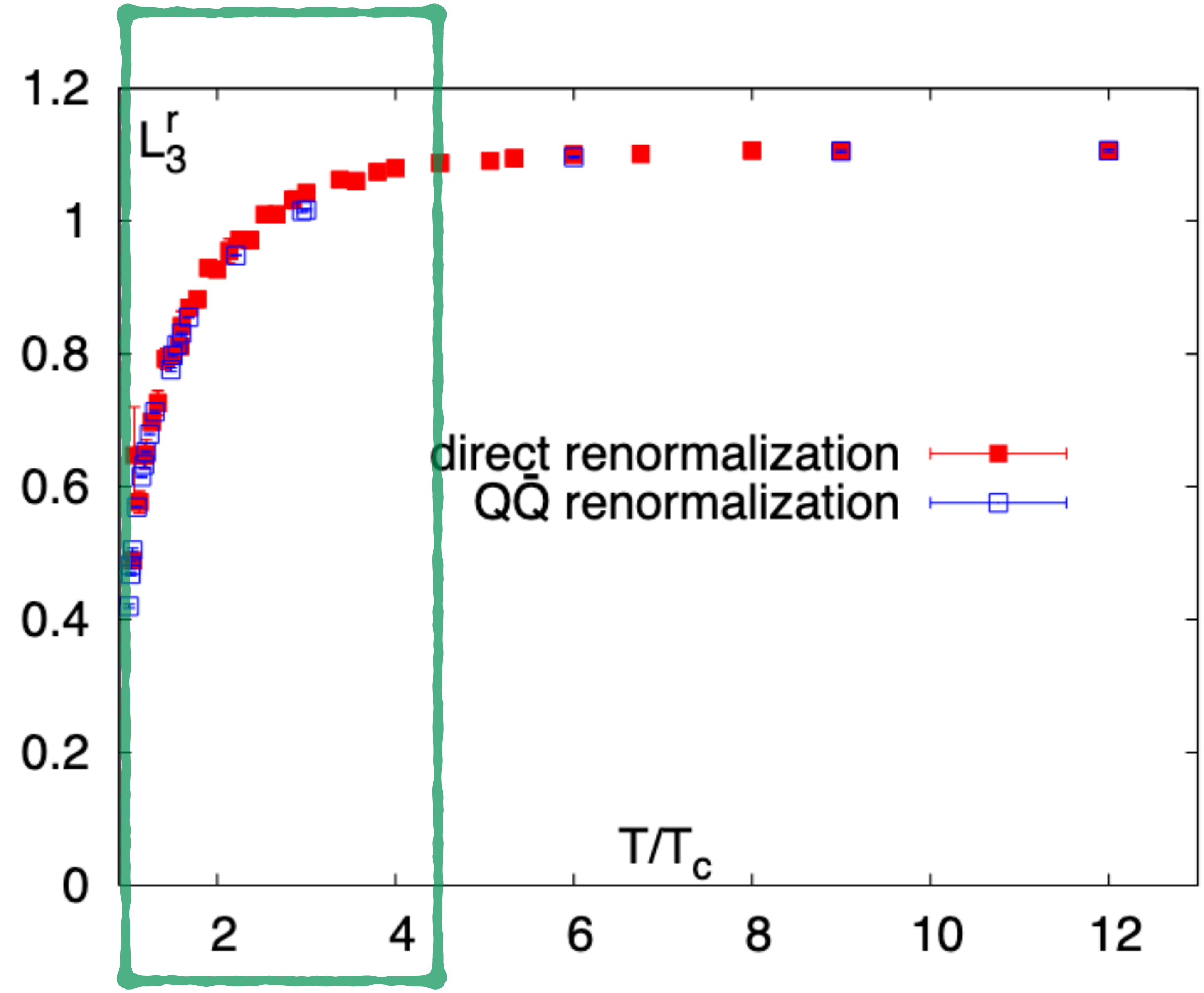
- Polyakov loop → an order parameter:
$\ell = 0$ in confined phase, $\ell = 1$ in deconfined phase.

Semi-QGP region



SU(3) Polyakov loop- an order parameter.[Ref: Gupta, Hubner and Kaczmarek, 0711.2251]

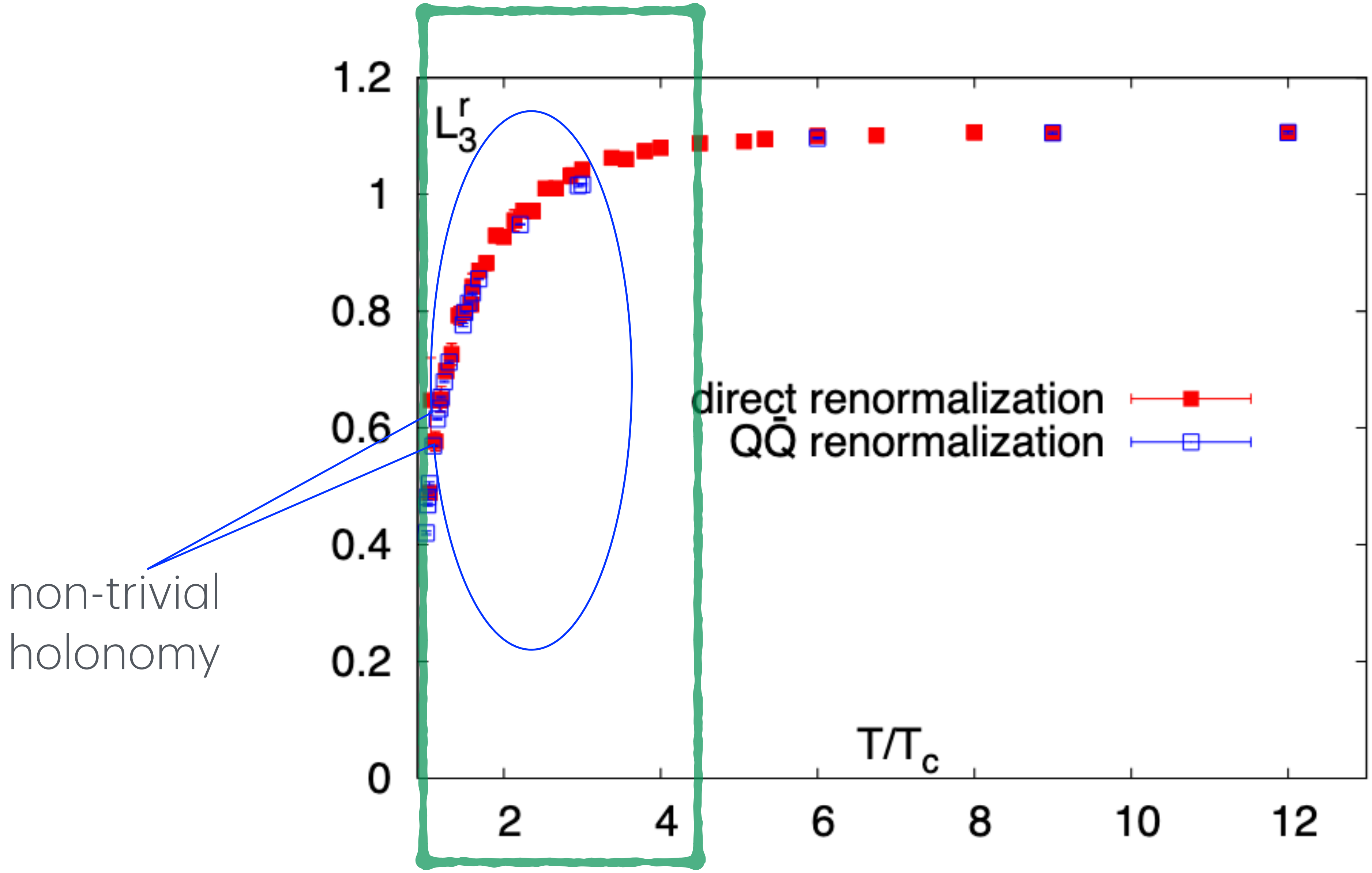
Semi-QGP region



SU(3) Polyakov loop- an order parameter.[Ref: Gupta, Hubner and Kaczmarek, 0711.2251]

1. Range: T_c to $\sim 4T_c$
2. $0.4 < \ell_1 < 1$ for $SU(3)$ gauge theory, without quarks. $\left(\ell_1 = \mathrm{Tr}L(\mathbf{x})\right)$

Semi-QGP region



SU(3) Polyakov loop- an order parameter.[Ref: Gupta, Hubner and Kaczmarek, 0711.2251]

To explain thermodynamics of this region of non-trivial holonomy, we need to add some potential due to non-zero holonomy or **Holonomous potential** to the free energy in the Semi-QGP region.

(our) Matrix Model

We can obtain the Free energy with non-trivial Holonomy in the semi-QGP region in our matrix model.

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- We add a background field to the gauge field. $A_{bg,\mu} = \frac{\mathbf{Q}}{g}\delta_{0,\mu} = \frac{2\pi T}{g}\mathbf{q}\,\delta_{0,\mu}$

$$L(\mathbf{x}) = \exp\left(\frac{2\pi i}{N_c}\mathbf{q}\right)$$

where $(\mathbf{q})_{ab} = q^a \delta_{ab}$ has $(N_c - 1)$ independent element, satisfying $\sum_{a=1}^{N_c} q^a = 0$

Ref: Dumitru, Guo, Hidaka, Altes, Pisarski; 1205:0137

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where $(\mathbf{q})_{ab} = q^a \delta_{ab}$ has $(N_c - 1)$ independent element, satisfying $\sum_{a=1}^{N_c} q^a = 0$

Hence the total field, $A_\mu = A_{bg,\mu} + B_\mu$

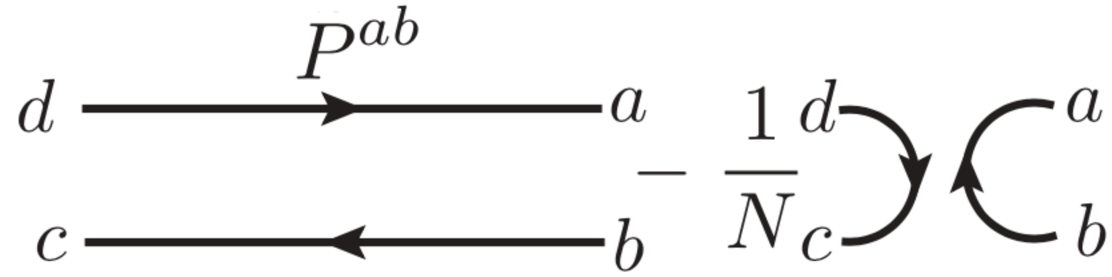
(B_μ is quantum gauge field, that is considered in the perturbative study.)

Ref: Dumitru, Guo, Hidaka, Altes, Pisarski; 1205:0137

Propagators

PRD 80, 036004 (2009) Pisarski & Hidaka

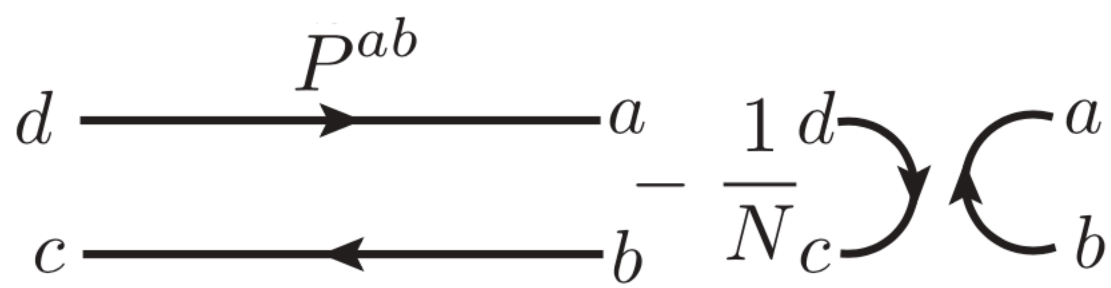
- Gluon Propagator:



$$\left\langle B_\mu^{ab}(P) B_\nu^{cd}(-P) \right\rangle = \left(\delta_{\mu\nu} - (1-\xi) \frac{P_\mu^{ab} P_\nu^{ab}}{\left(P^{ab}\right)^2} \right) \frac{1}{\left(P^{ab}\right)^2} \mathscr{P}^{ab,cd}$$

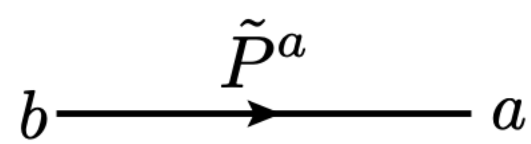
where $\mathscr{P}^{ab,cd} = \delta^{ac}\delta^{bd} - \frac{1}{N_c}\delta^{ab}\delta^{cd}$ and $P_\mu^{ab} = (p_0 + 2\pi(q^a - q^b), \mathbf{p}) = (p_0^{ab}, \mathbf{p})$

- Ghost Propagator:



$$\left\langle \eta^{ab}(P) \bar{\eta}^{cd}(-P) \right\rangle = \frac{1}{\left(P^{ab}\right)^2} \mathscr{P}^{ab,cd}$$

- Quark Propagator:



$$\left\langle \psi^a(P) \bar{\psi}^b(-P) \right\rangle = \frac{\delta^{ab}}{-i\widetilde{\not{P}}^{a} + m}$$

(our) Matrix Model

Free energy computation to Leading order:

$$V_{pert} \sim \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ab} T^4 \left(|q^{ab}|^2 (1 - |q^{ab}|)^2 - \frac{1}{30} \right)$$

$$\mathscr{P}^{ab,cd} = \delta^{ac}\delta^{bd} - \frac{1}{N_c}\delta^{ab}\delta^{cd} \qquad\qquad q^{ab} = \mathrm{mod}(q^a - q^b, 1)$$

Ref: Dumitru, Guo, Hidaka, Altes, Pisarski; 1205:0137

(our) Matrix Model

Appears due to non-zero Holonomy

Free energy computation to Leading order:

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- Note: The term $\sim q^2(1-q)^2$ describes the semi-QGP region, but it has an unexpected $\sim q^3$ term, which causes an first order phase transition. But there is no phase transition in this region.

Ref: Dumitru, Guo, Hidaka, Altes, Pisarski; 1205:0137

(our) Matrix Model

... Story doesn't end here.

Even the Ward identity is not satisfied now! $P_\mu^{ab}\Pi_{\mu\nu}^{ab,cd} = \#\, iu_\nu \neq 0$

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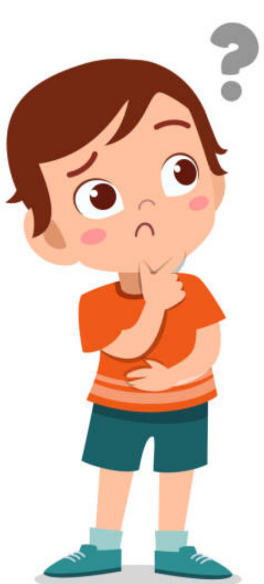
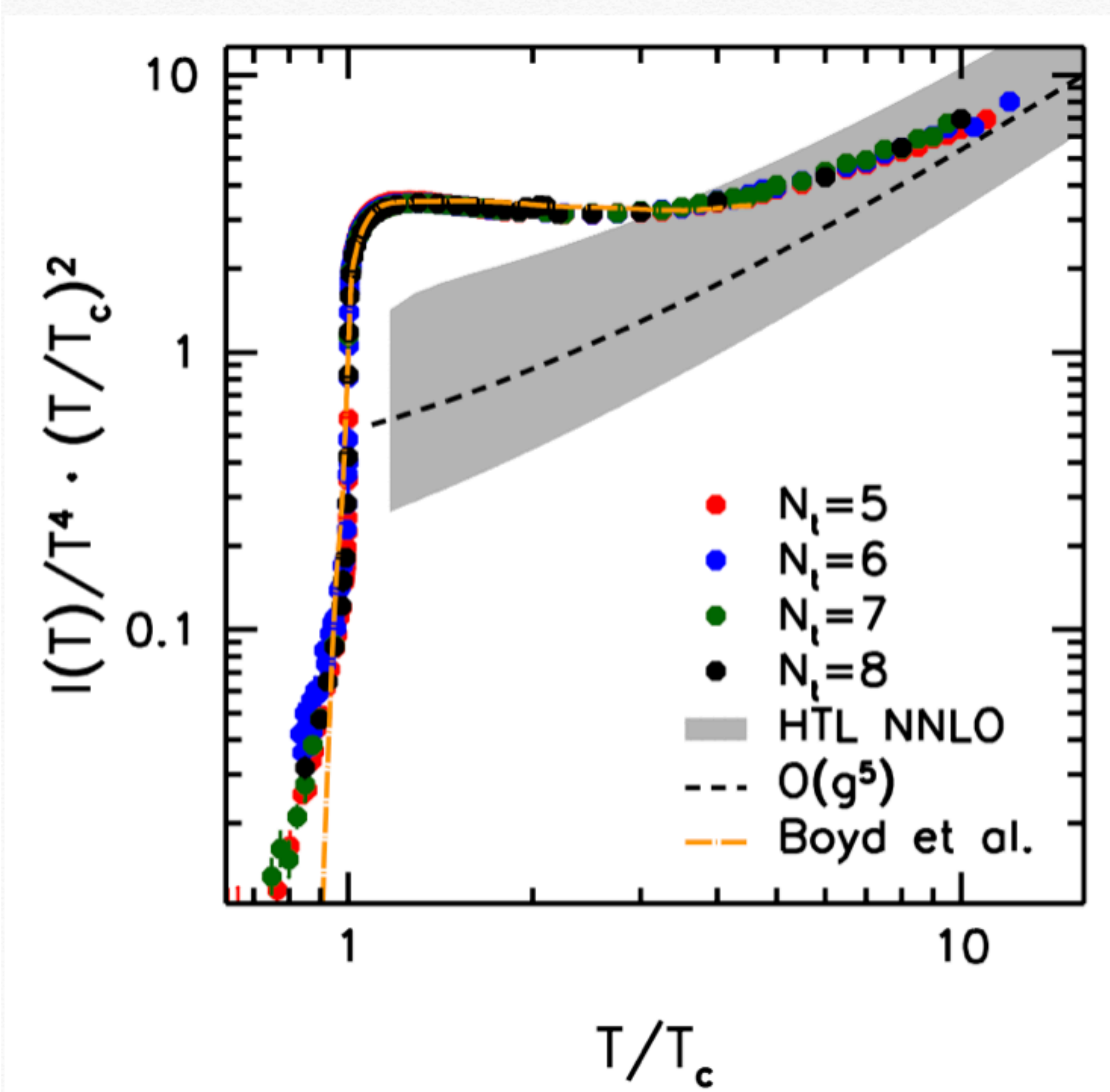
Seems like introduction of Background field characterized by Polyakov loop creates more problem rather than solving.

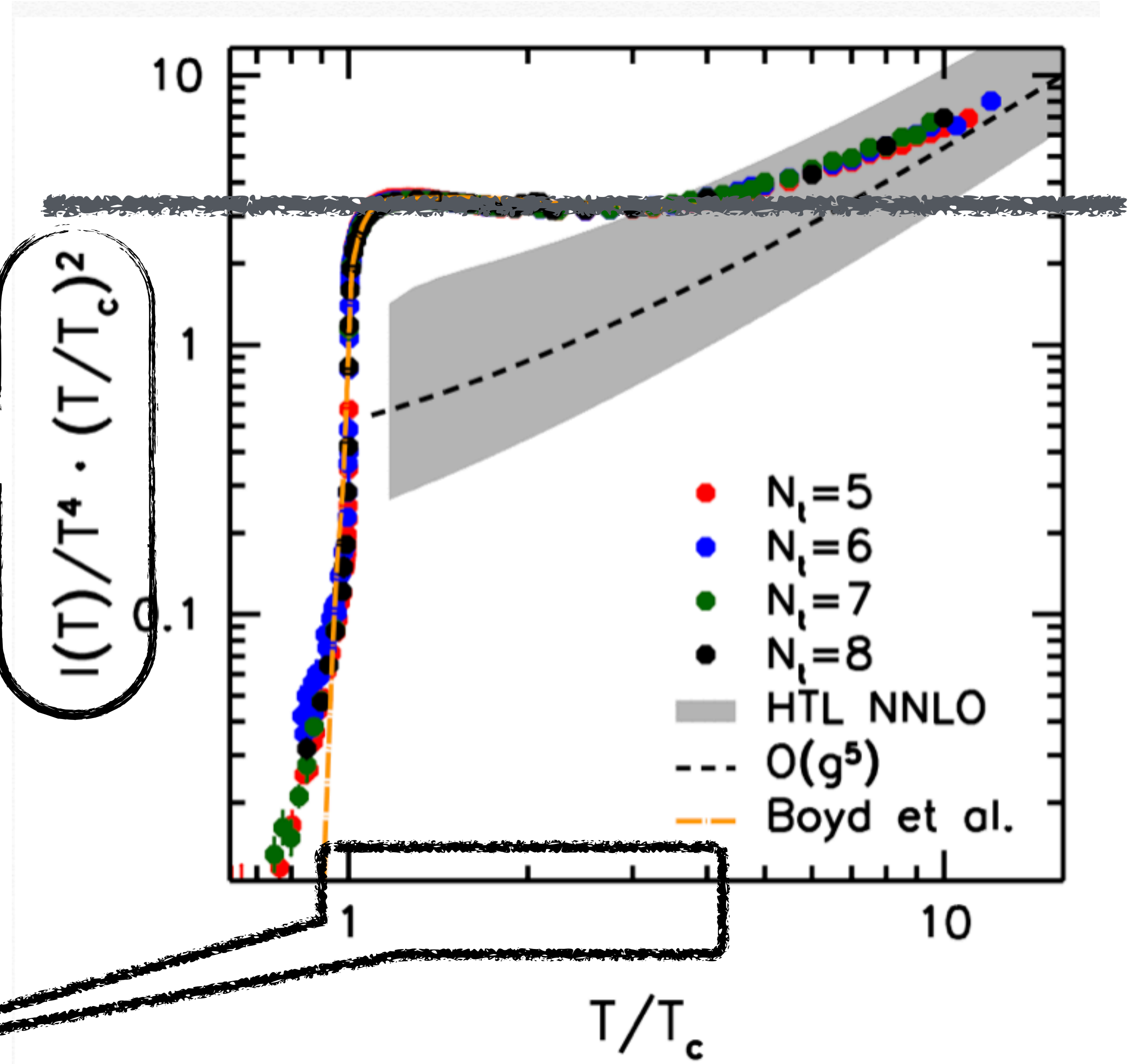


Interaction measure



Interaction Measure. [Ref: Borsanyi, Endrodi, Fodor, Katz, Szabo, 1204.6184]

Interaction measure



$$\frac{I}{T^2 T_c^2}$$

$1.2\,T_c \to 4\,T_c$

Interaction Measure. [Ref: Borsanyi, Endrodi, Fodor, Katz, Szabo, 1204.6184]

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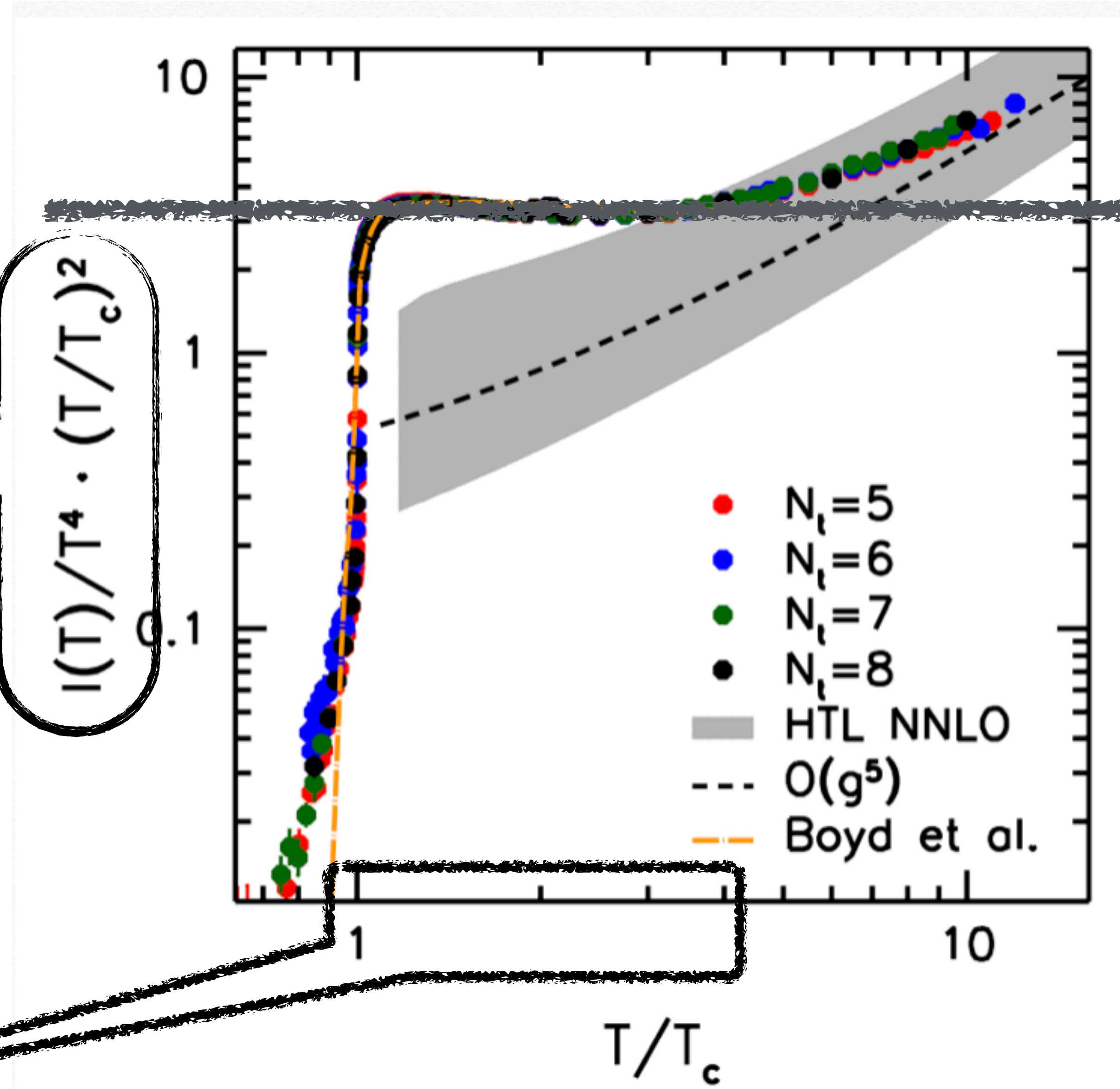

$1.2\,T_c \rightarrow 4\,T_c$

Interaction Measure. [Ref: Borsanyi, Endrodi, Fodor, Katz, Szabo, 1204.6184]

In $1.2\,T_c \rightarrow 4\,T_c$, leading correction to pressure in pure gauge theory, behaves $\sim T^2$. Not exact, but approximately.

Free-energy

$$V_{pert} \sim \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ab} T^4 \left(|q^{ab}|^2 (1 - |q^{ab}|)^2 - \frac{1}{30} \right)$$

In addition to the above Free energy, we need a non-perturbative term to the free energy that behaves $\sim T^2$.

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So, we need such a term, $V_{non-pert} \sim \sum_{a,b} \mathscr{P}^{ab,ab}\, CT^2 \left(-q^{ab}(1 - q^{ab}) + \frac{1}{6} \right)$

(our) Matrix Model- Teen Field (৩)

- Pisarski and Hidaka suggested that, we can consider a dynamical field that can generate non-perturbative term in the Free energy. [Hidaka, Pisarski, arXiv: 2009.03903]
- It has to be Two dimensional in nature, that is, its momentum is suppressed in two spatial directions to give $\sim T^2$ contribution in the free energy.
- It can be represented in adjoint representation and it is a ghost field.

We named it "Teen" field (notation for later use: ৩)

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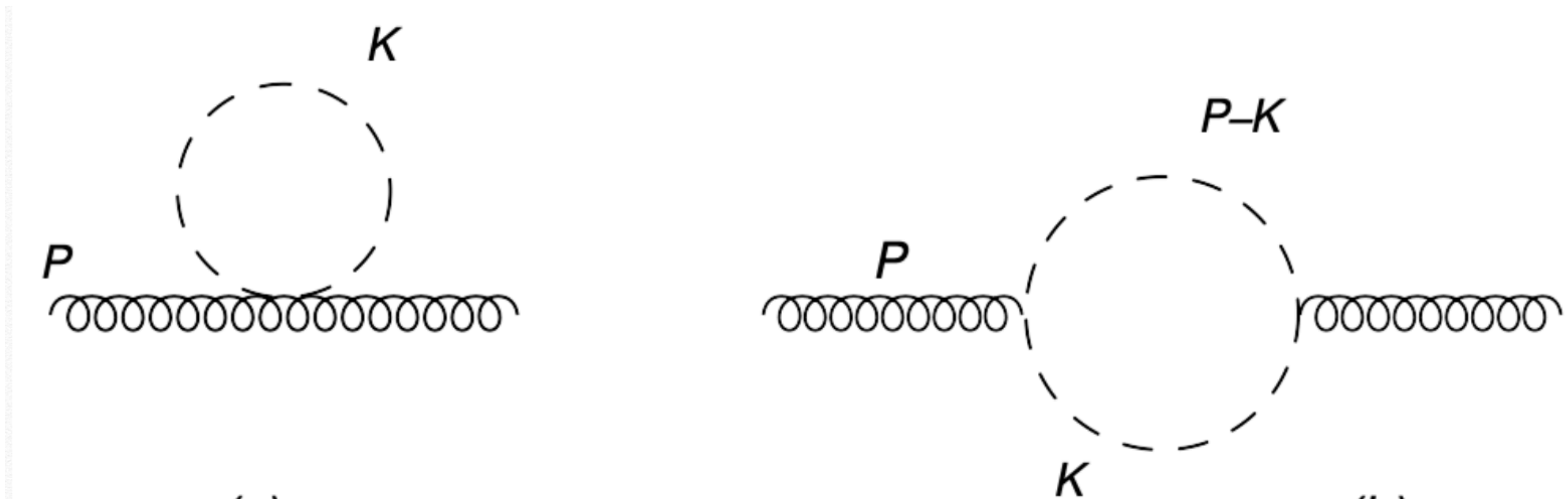
- ৩ (Teen) field propagator in momentum space,

$$\Delta^{ab}_{৩}(p_0^{ab}, \mathbf{p}) = \frac{1}{(p_0^{ab})^2 + p_{||}^2 + p_{\perp}^2}$$

where $p_{\perp} \leq T_c$ and $T_c \ll gT \ll T$.

Teen Field (ও) self-energy

- It also solves the problem of Ward identity



Teen field contribution to gluon self energy

Now, the total self energy:

$$\Pi_{total}^{\mu\nu;ab,cd} = \Pi_{gl}^{\mu\nu;ab,cd} + \Pi_{\text{ও}}^{\mu\nu;ab,cd} = -(m^2)^{ab,cd}\delta\Pi^{\mu\nu}(P^{ab})$$

which satisfies Ward identity, since, $P_\mu^{ab}\delta\Pi_{\mu\nu} = 0$

Kinetic theory and Transport Coefficient

- We use Kinetic theory in the semi-QGP region, that is when $q^{ab} \neq 0$ for gluons and teens.

$$\mathcal{S}_{\text{ব}}\, 2P^{\mu,a_1 b_1} \partial_\mu f_{\text{ব},a_1 b_1}(\boldsymbol{x}, \boldsymbol{p}, t) = -\mathcal{C}_{\text{ব},a_1 b_1}[f_{\text{ব}}]$$

ব= gluon, teen and $\mathcal{S}_{\text{gluon}} = 1,\ \mathcal{S}_{\text{teen}} = -1$

- Teen is equivalent to a fermion with an imaginary chemical potential $i\pi T$, which corresponds to the replacement of the fermion distribution function f_f with $-f_{\text{ঙ}}$.

$$\frac{1}{\mathrm{e}^{(E_p - i(Q^a - Q^b + \pi T))/T} + 1} = -\frac{1}{\mathrm{e}^{(E_p - i(Q^a - Q^b))/T} - 1} = -f^0_{ab}\,.$$

$$f^0_{\mathrm{g},ab} = f^0_{ab}(E_p) \equiv \frac{1}{\mathrm{e}^{(E_p - i(Q^a - Q^b))/T} - 1}, \qquad f^0_{\text{ঙ},ab}(E_p) = f^0_{\mathrm{g},ab}(E_p)$$

- The sign before teen distribution function is taken care using the sign function $\mathcal{S}$.

- Energy-Momentum Tensor:

$$T_{\mathrm{g}}^{\mu\nu}(\boldsymbol{r},t) = 2\int_{\mathrm{g}} d\Gamma_{ab}\; P^{\mu,ab}P^{\nu,ab}\; f_{\mathrm{g},ab}(\boldsymbol{r},\boldsymbol{p},t)$$

$$T_{\circlearrowleft}^{\mu\nu} = 2\int_{\circlearrowleft} d\Gamma_{ab}\; P^{\mu,ab}P^{\nu,ab}\; \mathcal{S}_{\circlearrowleft}\; f_{\circlearrowleft,ab}\;.$$

- Decomposition of $T^{\mu\nu} = \mathscr{E} u^{\mu}u^{\nu} + \mathscr{P}\Delta^{\mu\nu} + \Pi^{\mu\nu}$

$$\Pi^{\mu\nu} = \eta\,\sigma^{\mu\nu} + \zeta\,\Delta^{\mu\nu}\Delta^{\alpha\beta}\partial_{\alpha}u_{\beta} + \cdots$$

Collision term!

$$\mathcal{C}_{\text{ব},a_1b_1}[f_{\text{ব}}] = \frac{1}{2}\sum_{\text{দ},\text{ব}',\text{দ}'} \mathcal{S}_{\text{ব}}\,\mathcal{S}_{\text{দ}} \int_{\text{দ}} d\Gamma_{a_2b_2} \int_{\text{ব}'} d\Gamma_{a_3b_3} \int_{\text{দ}'} d\Gamma_{a_4b_4}\,(2\pi)^4\delta^{(4)}(P_1^{a_1b_1} + P_2^{a_2b_2} - P_3^{a_3b_3} - P_4^{a_4b_4})$$

$$\times|\mathcal{M}_{\text{বদ}\to\text{ব}'\text{দ}'}|^2\left[f_{\text{ব},a_1b_1}f_{\text{দ},a_2b_2}(1+f_{\text{ব}',a_3b_3})(1+f_{\text{দ}',a_4b_4}) - f_{\text{ব}',a_3b_3}f_{\text{দ}',a_4b_4}(1+f_{\text{ব},a_1b_1})(1+f_{\text{দ},a_2b_2})\right].$$

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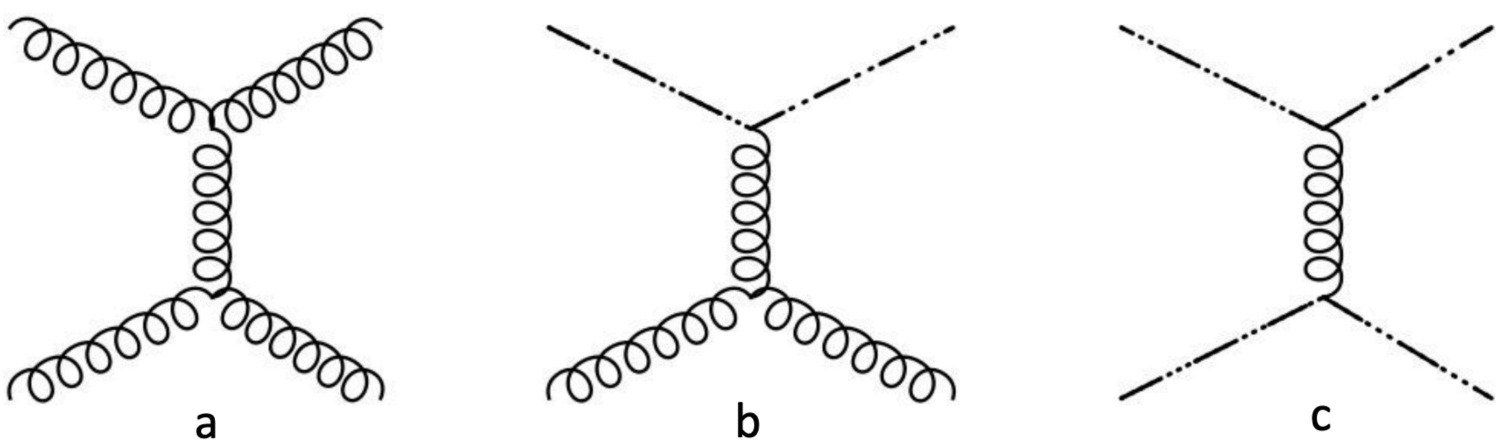
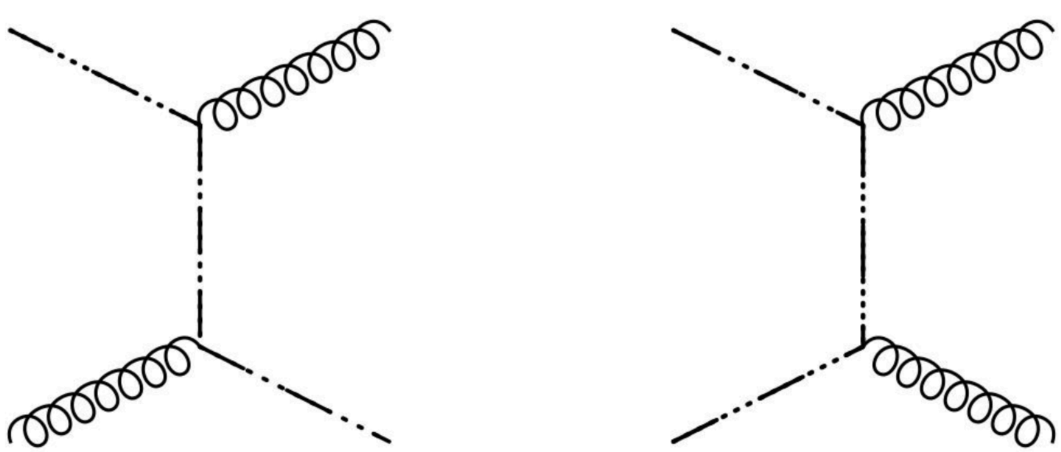
$$\times\boxed{|\mathcal{M}_{\text{বদ}\to\text{ব}'\text{দ}'}|^2}\Big[f_{\text{ব},a_1b_1}f_{\text{দ},a_2b_2}(1+f_{\text{ব}',a_3b_3})(1+f_{\text{দ}',a_4b_4}) - f_{\text{ব}',a_3b_3}f_{\text{দ}',a_4b_4}(1+f_{\text{ব},a_1b_1})(1+f_{\text{দ},a_2b_2})\Big]$$



Contributes at leading order



Does not contribute at leading order

Following a process, similar to AMY's (*JHEP* 11 (2000) 001), We expand the distribution function around the distribution function in thermal equilibrium as,

$$f_{ব,ab} = f^0_{ab} + \epsilon f^1_{ব,ab} + \epsilon^2 f^2_{ব,ab} + \cdots \ .$$

Where $f^n_{ব,ab} = \ f^0_{ab}(1 + f^0_{ab})\phi^{(n)}_{ব,ab}$

Note: For the current calculation, we take only first order correction in the distribution function.

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Note: For the current calculation, we take only first order correction in the distribution function.

From collision term we introduce the linearized operator,

$$(\mathcal{L}\phi^{(n)})_{ব,a_1b_1} = \frac{1}{2}\sum_{দ,ব',দ'} \mathcal{S}_{ব}\mathcal{S}_{দ}\int_{দ} d\Gamma_{a_2b_2}\int_{ব'} d\Gamma_{a_3b_3}\int_{দ'} d\Gamma_{a_4b_4}\,(2\pi)^4\delta^{(4)}(P_1^{a_1b_1} + P_2^{a_2b_2} - P_3^{a_3b_3} - P_4^{a_4b_4})$$
$$\times|\mathcal{M}_{বদ\to ব'দ'}|^2\left[f^0_{a_1b_1}f^0_{a_2b_2}(1 + f^0_{a_3b_3})(1 + f^0_{a_4b_4})\right]\left(\phi^{(n)}_{ব,a_1b_1} + \phi^{(n)}_{দ,a_2b_2} - \phi^{(n)}_{ব',a_3b_3} - \phi^{(n)}_{দ',a_4b_4}\right)$$

To leading order, the viscosities are computed from,

$$\mathcal{S}_{ব}\, 2P^{\mu,a_1,b_1}\partial_\mu^{(1)} f_{ব,a_1b_1} = -(\mathcal{L}\phi^{(1)})_{ব,a_1b_1}$$

For shear viscosity, LHS becomes, $2P^{\mu,a_1b_1}\partial_\mu^{(1)} f_{ব,a_1b_1} = -\beta f^0_{a_1b_1}(1+f^0_{a_1b_1})p^ip^j\sigma_{ij}$

with $\sigma_{ij} = \partial_i u_j + \partial_j u_i - \frac{2}{3}\delta^{ij}\partial_k u_k$.

- We parameterized $\phi^{(1)}_{ব,ab}$ as $\phi^{(1)}_{ব,ab} = \sigma_{ij}\left(p^ip^j - \frac{1}{3}p^2\delta^{ij}\right)\chi_{ব,ab}(p) = \sigma_{ij}\chi^{ij}_{ব,ab}(p)$

So the dissipative part of the energy momentum tensor gives,

$$\delta T^{ij} = \sum_{ব} 2\mathcal{S}_{ব}\int_{ব} d\Gamma_{ab}\, p^ip^j f^{(1)}_{ব,ab} \quad = \sigma^{ij}\frac{2T}{5}\sum_{ব}\int_{ব} d\Gamma_{ab}\chi^{kl}_{ব,ab}(\mathcal{L}\chi^{kl})_{ব,ab}$$

$$\delta T^{ij} = \eta\, \sigma^{ij}$$

$$\eta = \frac{2T}{5} \sum_{\text{ব}} \int_{\text{ব}} d\Gamma_{ab}\, \chi^{ij}_{\text{ব},ab} (\mathcal{L}\chi^{ij})_{\text{ব},ab}$$

Expansion: $\chi_{\mathrm{g},ab} = \sum_{n=0}^{\infty} c_n^{\mathrm{g}}\, \chi_n^{\mathrm{g}}(p)$ and $\chi_{\text{৩},ab} = \sum_{n=0}^{\infty} c_n^{\text{৩}}\, \chi_n^{\text{৩}}(p)$

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After some computation:

$$\eta = \frac{4\pi^5 T^3}{225(g(T)^2 N_c)^2 N_c^2 \ln(\kappa/g^2 N_c)} \left[\frac{(d_0^{\text{৩}})^2}{\chi_{00}^{\text{৩৩}} + \frac{T^2}{T_d^2}\chi_{00}'^{\text{৩৩}}} + \frac{(d_0^{\mathrm{g}})^2}{\chi_{00}^{\mathrm{gg}} + \frac{T_d^2}{T^2}\chi_{00}'^{\mathrm{gg}}} \right]$$

$$\eta = \frac{4\pi^5 T^3}{225(g(T)^2 N_c)^2 N_c^2 \ln(\kappa/g^2 N_c)} \left[\frac{(d_0^{\circlearrowright})^2}{\chi_{00}^{\circlearrowright\circlearrowright} + \frac{T^2}{T_d^2}\chi_{00}^{\prime\circlearrowright\circlearrowright}} + \frac{(d_0^{\mathrm{g}})^2}{\chi_{00}^{\mathrm{gg}} + \frac{T_d^2}{T^2}\chi_{00}^{\prime\mathrm{gg}}} \right]$$

Each elements of this part depends on T/T_d, and various moments of the Polyakov loop, ℓ_k

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Each elementsThis part depends on T/T_d, and various moments of the Polyakov loop, ℓ_k

$$\ell_k = \int_{-q_0}^{+q_0} dq\, \rho(q)\cos(2\pi k q)$$

$$\rho(q) = 1 + b\cos(d\,q)$$

(Eigen value density)

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$$\rho(q) = 1 + b\cos(d\,q)$$

(Eigen value density)

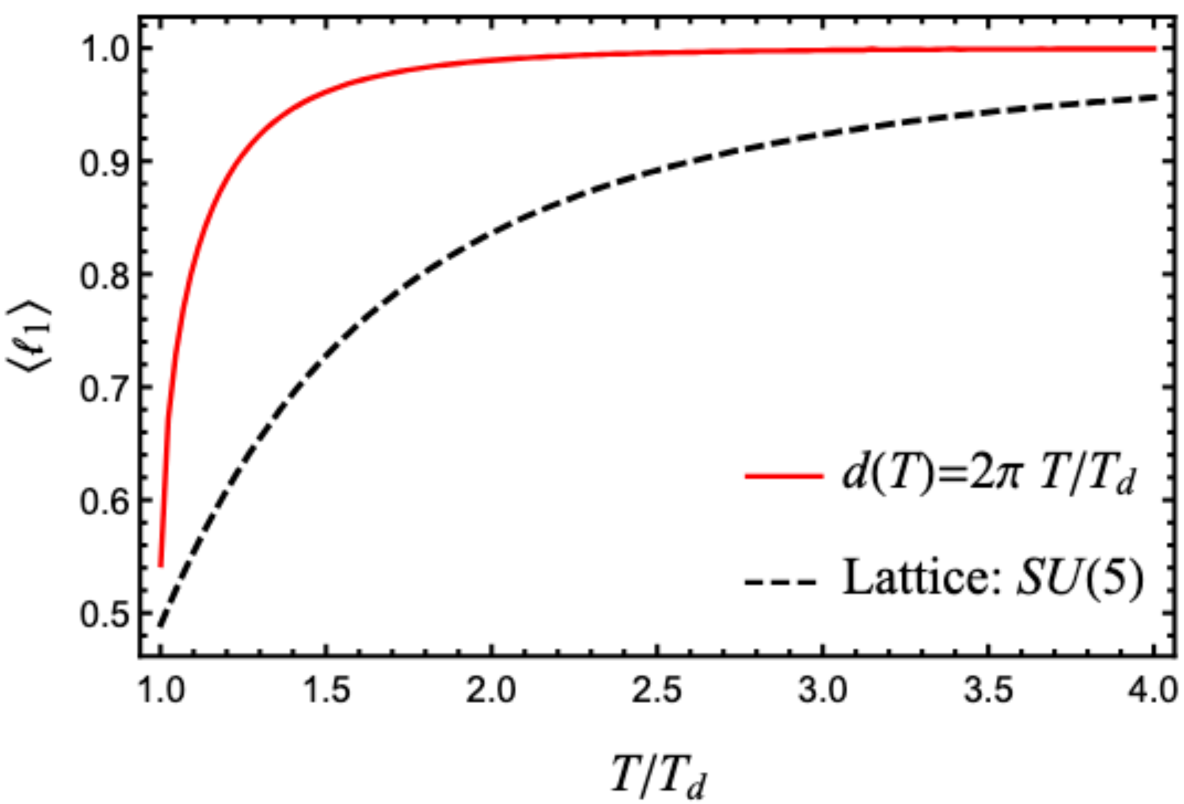
$$\cot(d\,q_0) = \frac{d}{3}\left(\frac{1}{2} - q_0\right) - \frac{1}{d(1/2 - q_0)}$$

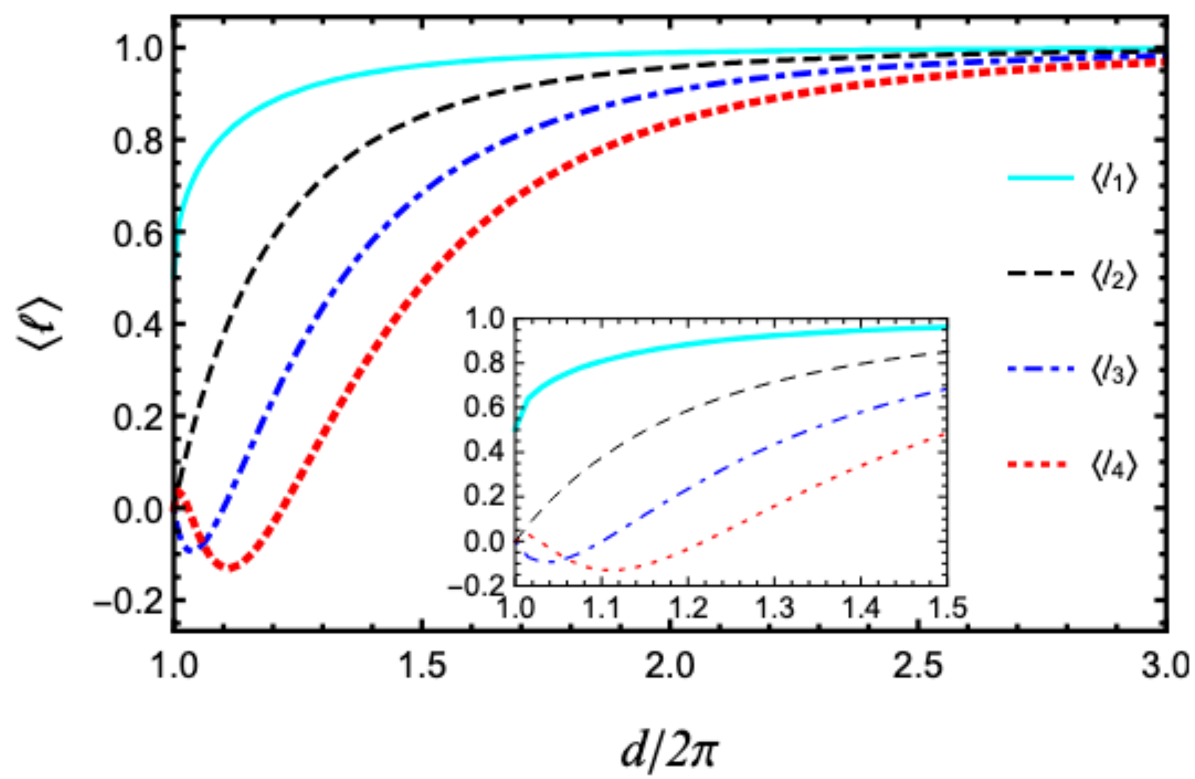
$$b^2 = \frac{d^4}{9}\left(\frac{1}{2} - q_0\right)^4 + \frac{d^2}{3}\left(\frac{1}{2} - q_0\right)^2 + 1$$

Loops!!

For the simplest ansatz, $d(T) = 2\pi\, T/T_d$, Polyakov loop at kth order,

$$\ell_k = \int_{-q_0}^{+q_0} dq\ \rho(q)\cos(2\pi k q)$$

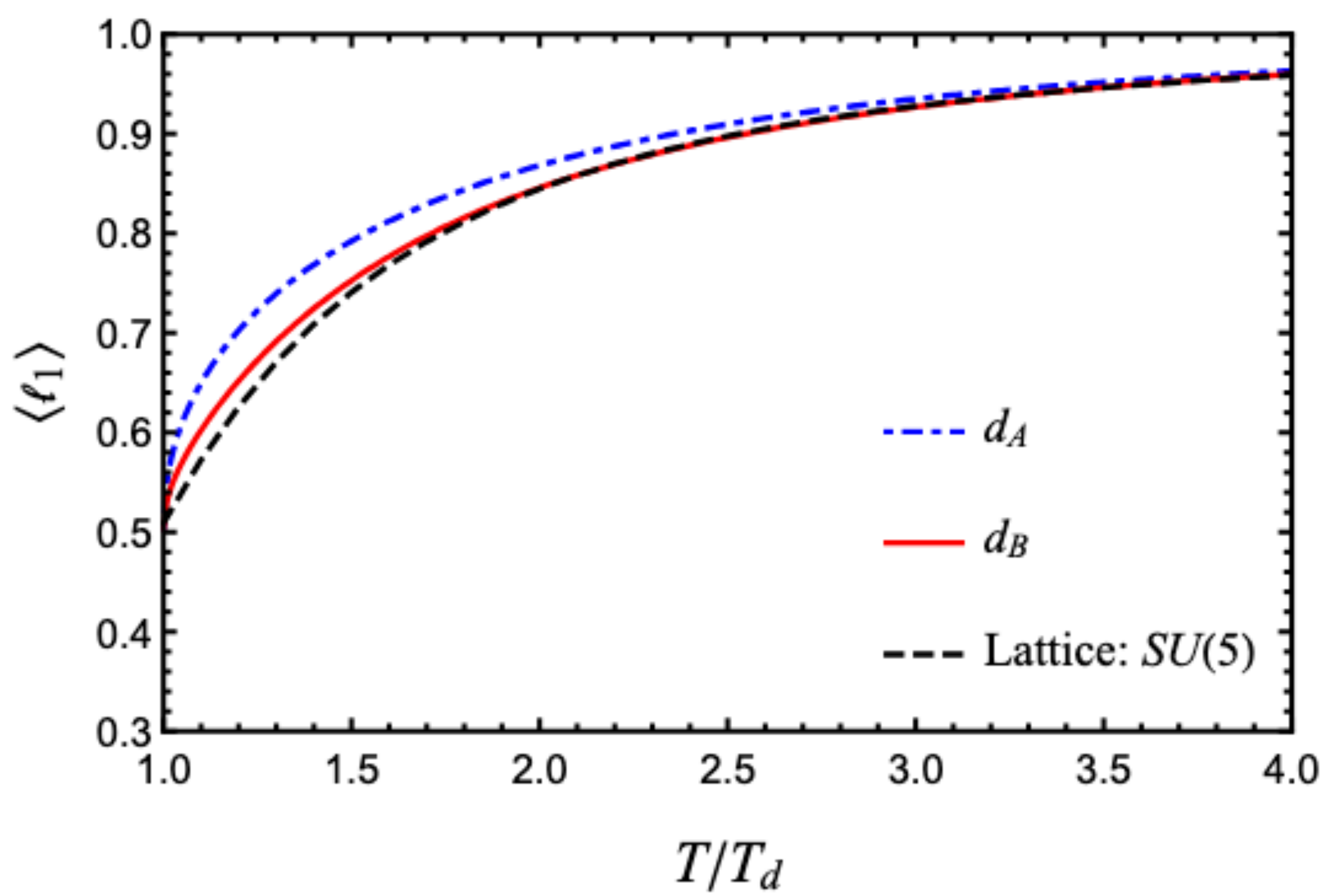




Loops!!

We tried another few ansatzs, but two best choice were,

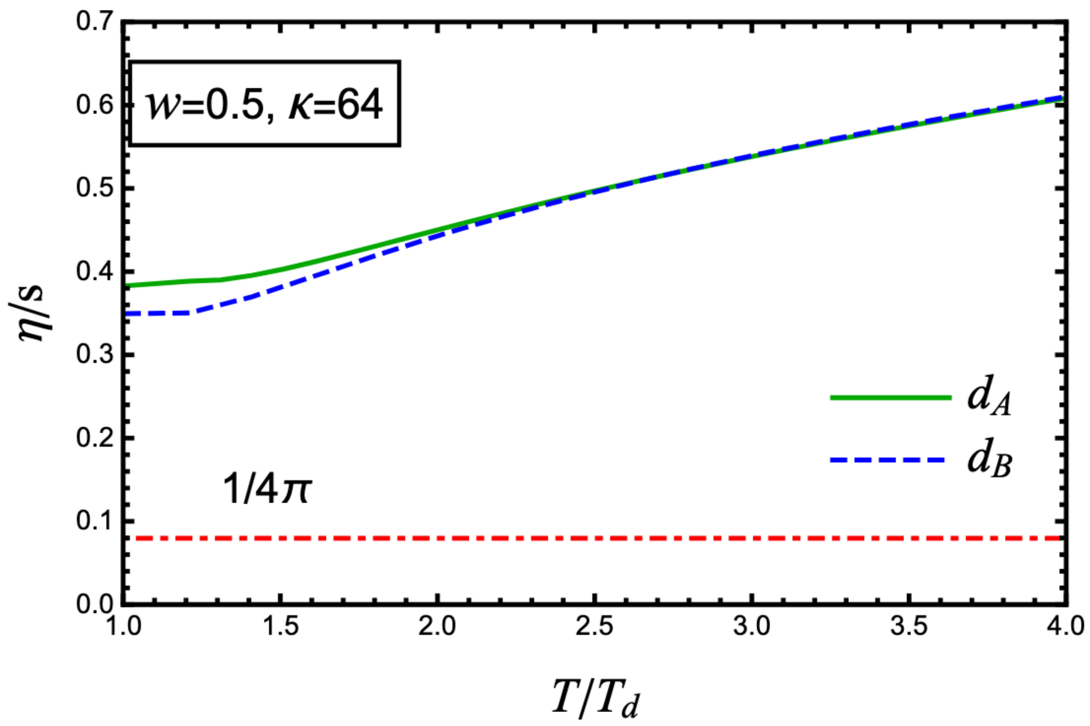
$$d_A(T) = 1.08\,t + 5.2032$$
$$d_B(T) = \frac{0.26}{t^3} + 1.105\,t + 4.9182\ ,\ t = \frac{T}{T_d}$$

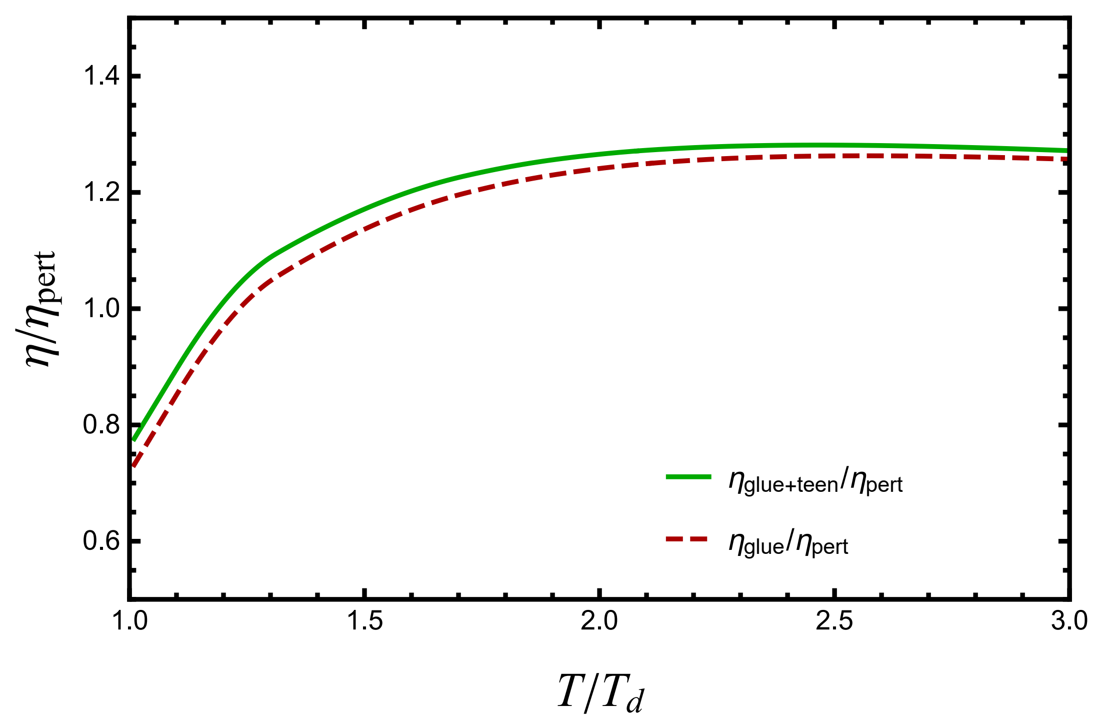


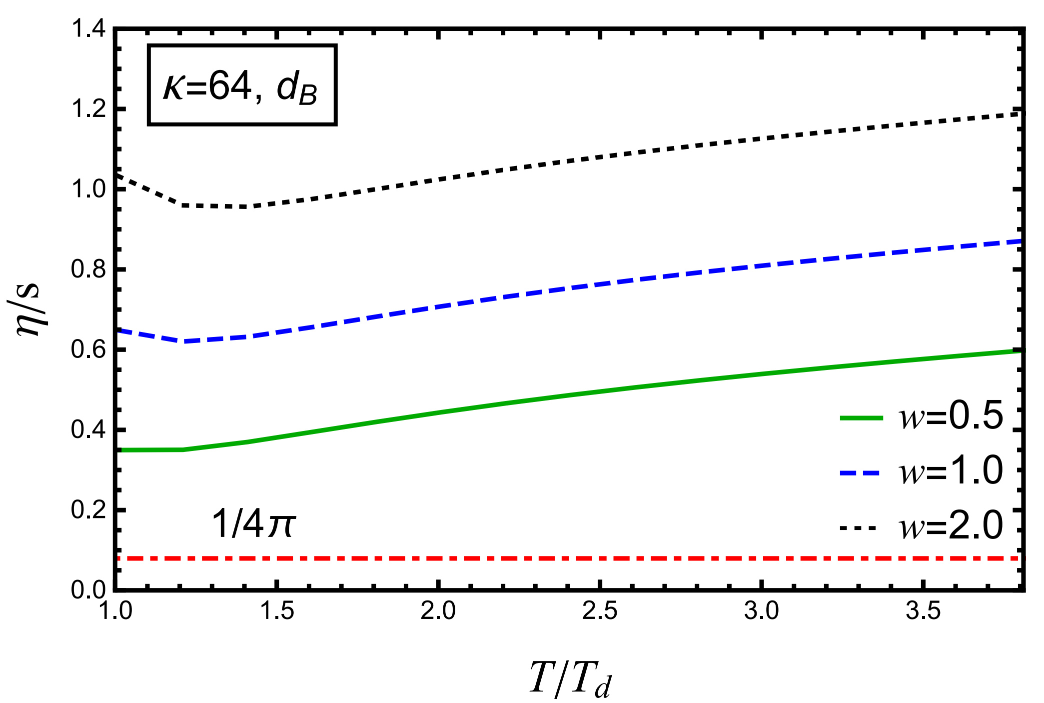
Result- Shear Viscosity

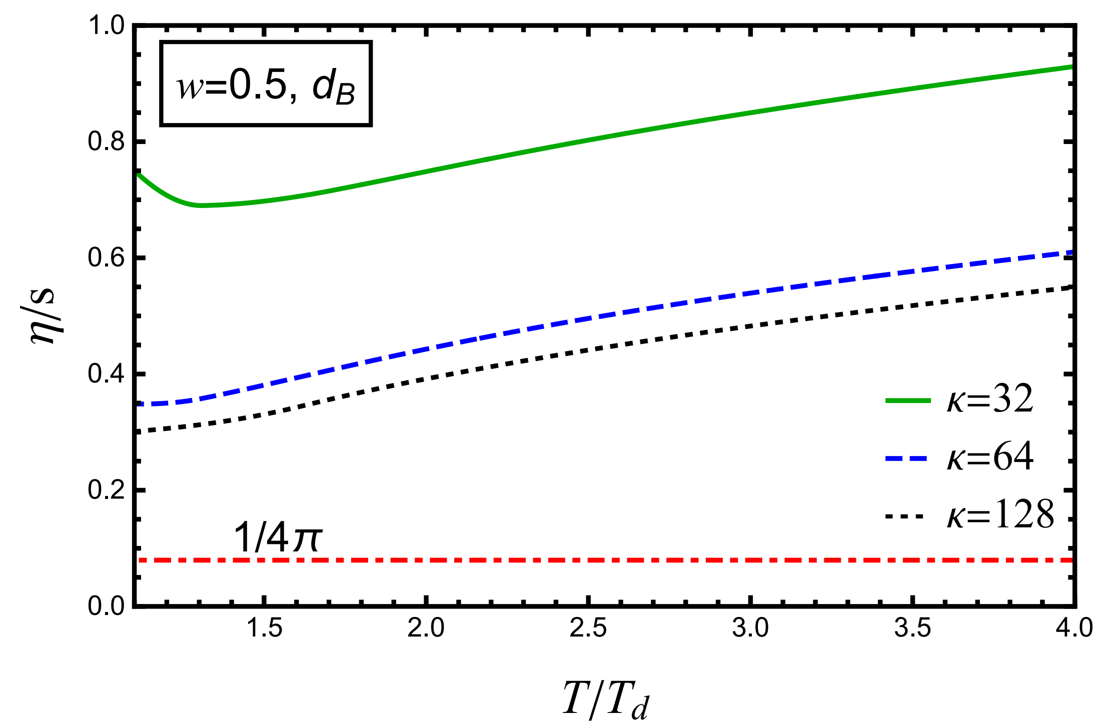
NH, R Ghosh, M Debnath, Y Hidaka, R Pisarski (in preparation)

$$\eta = \frac{T^3}{g^4 \log(\kappa/g^2 N_c)} F(T/T_d)$$







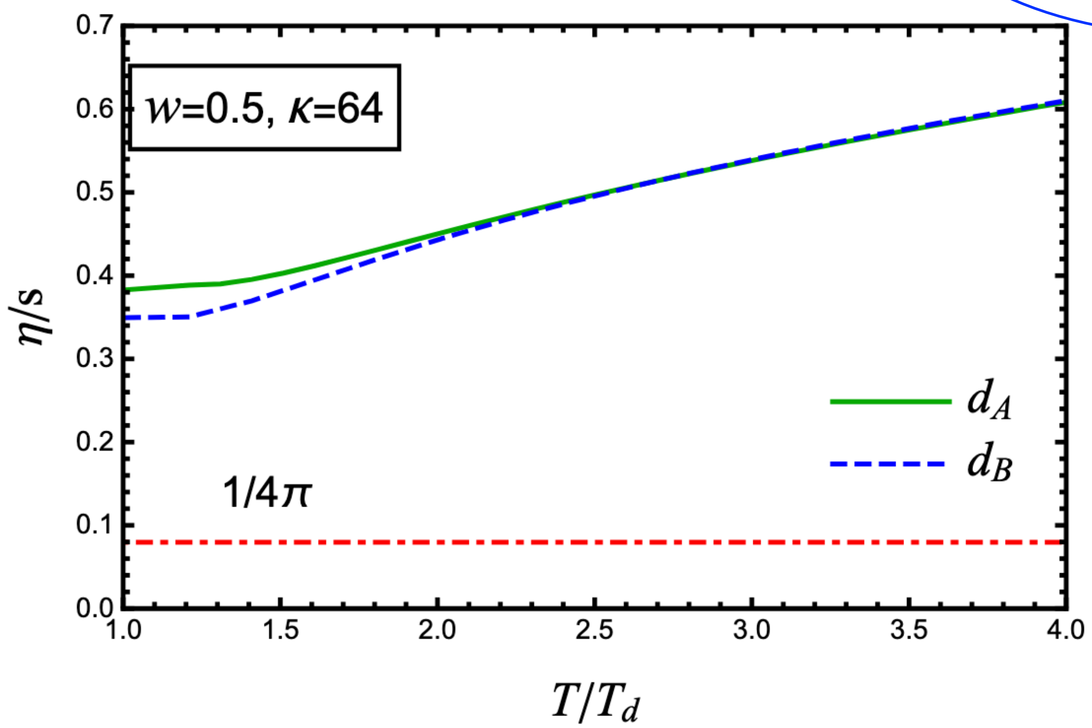


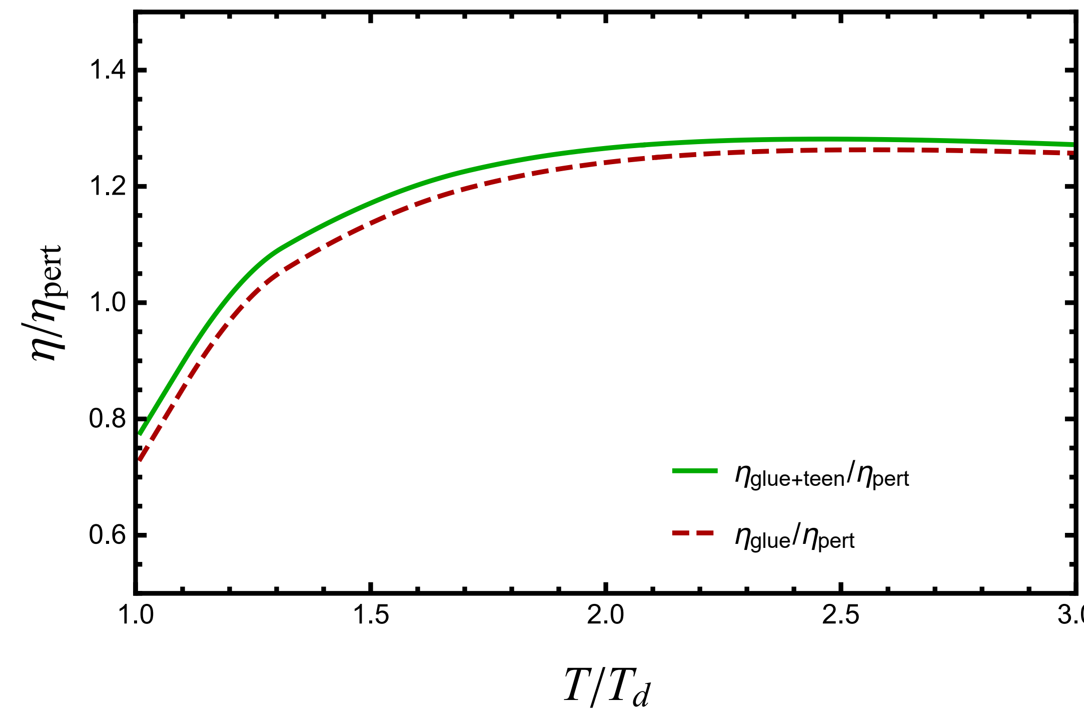
Result- Shear Viscosity

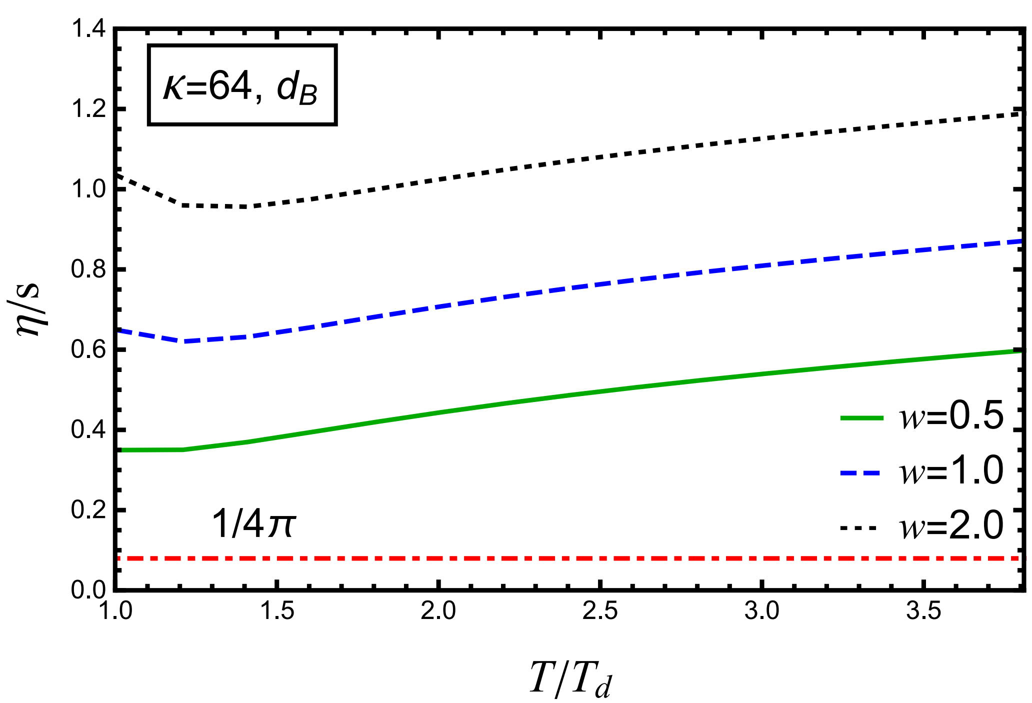
NH, R Ghosh, M Debnath, Y Hidaka, R Pisarski (in preparation)

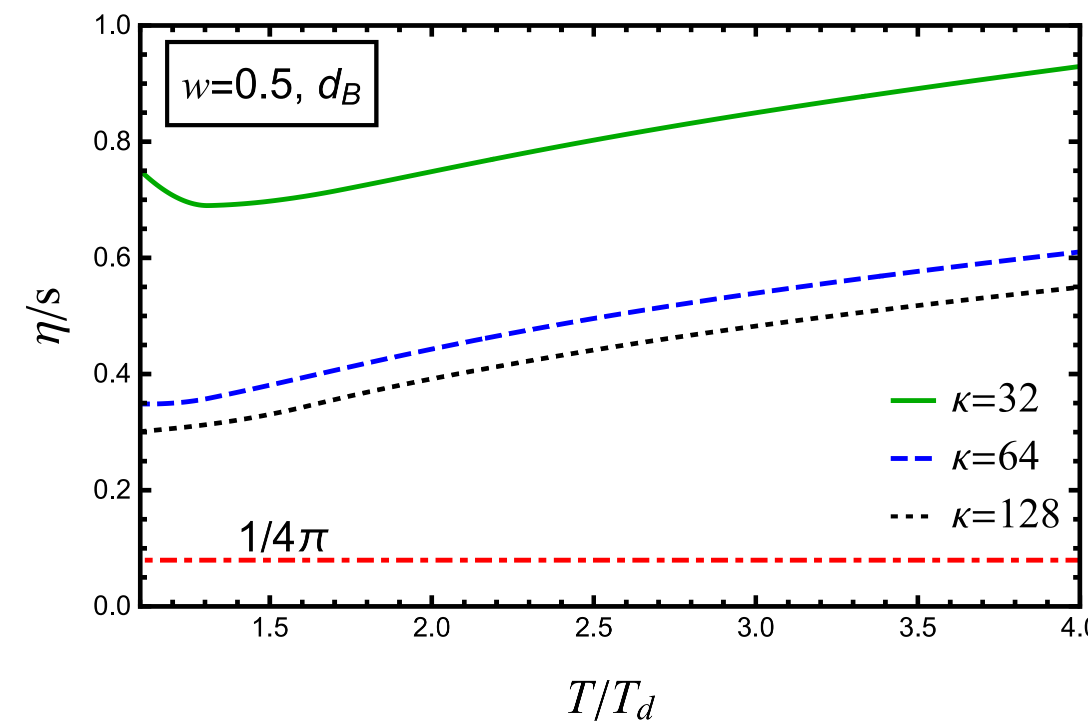
Appears in perturbative computation

$$\eta = \frac{T^3}{g^4 \log(\kappa/g^2 N_c)} F(T/T_d)$$





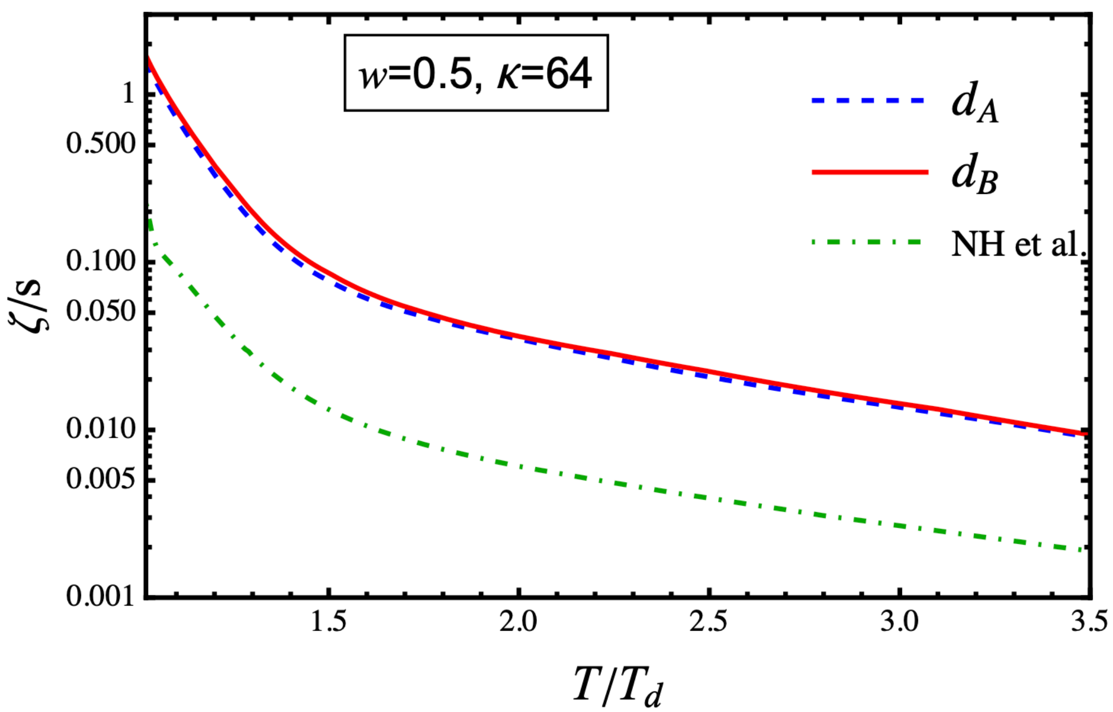


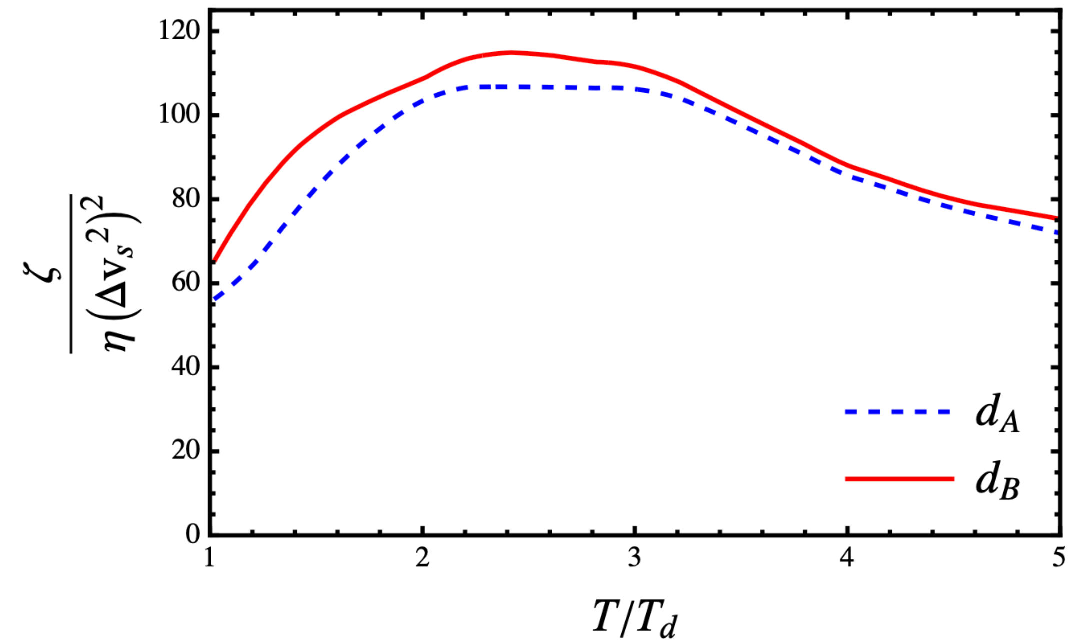


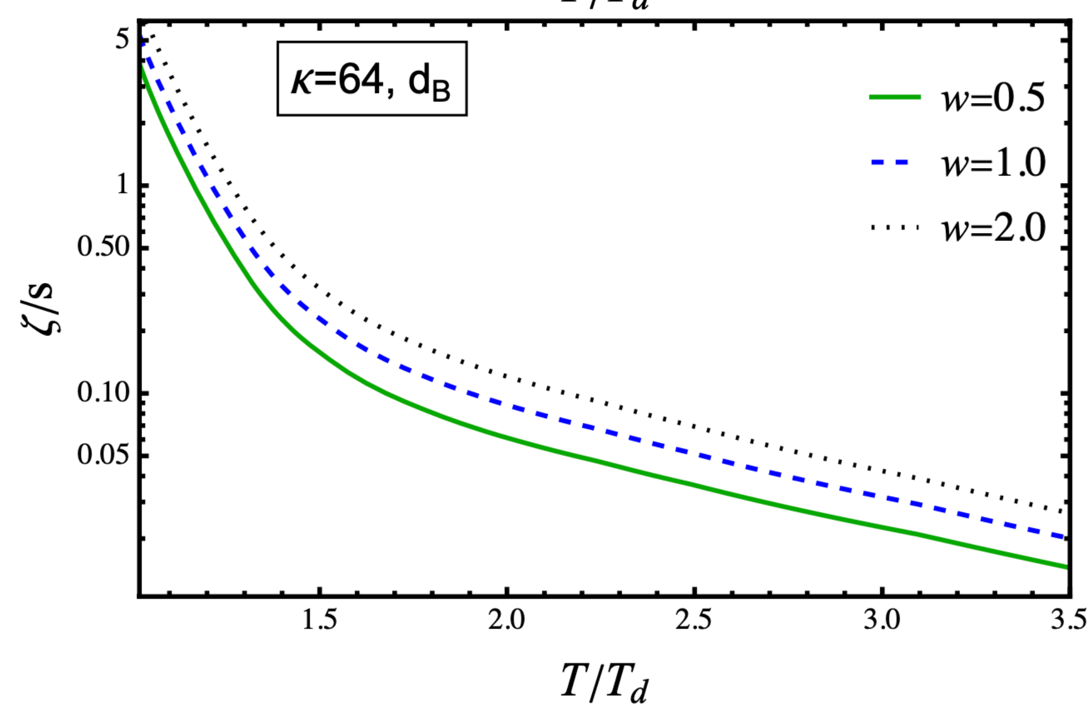
Result- Bulk Viscosity

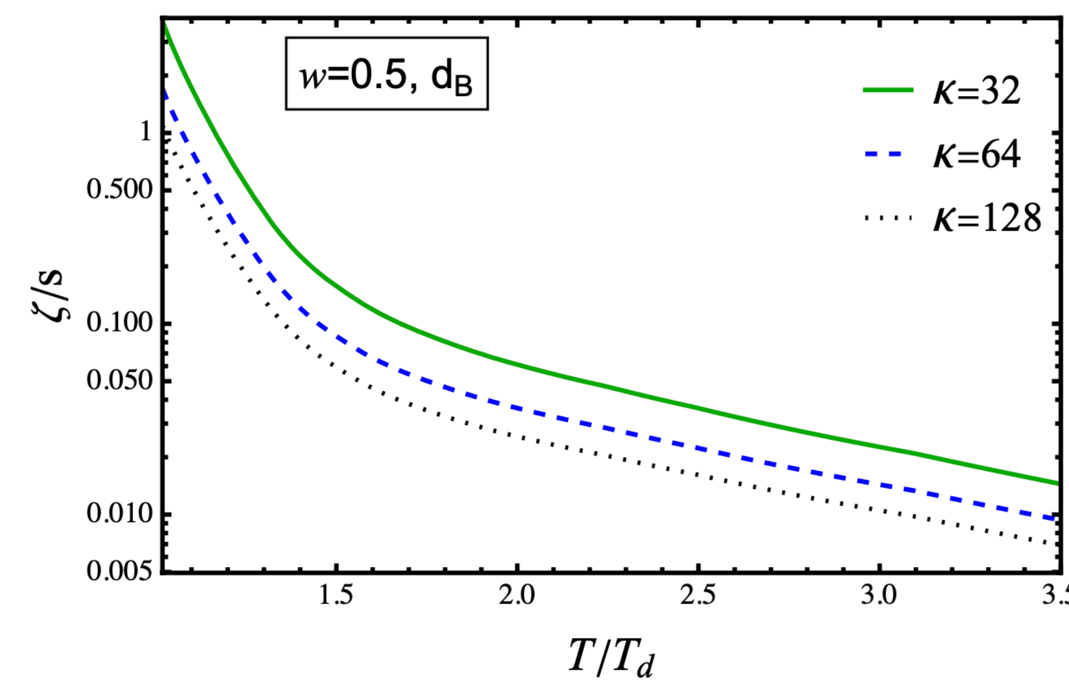
NH, R Ghosh, M Debnath, Y Hidaka, R Pisarski (in preparation)

$$\zeta = \frac{T^3(\Delta v_s^2)^2}{g^4 \log(\kappa/g^2 N_c)} G(T/T_d)$$







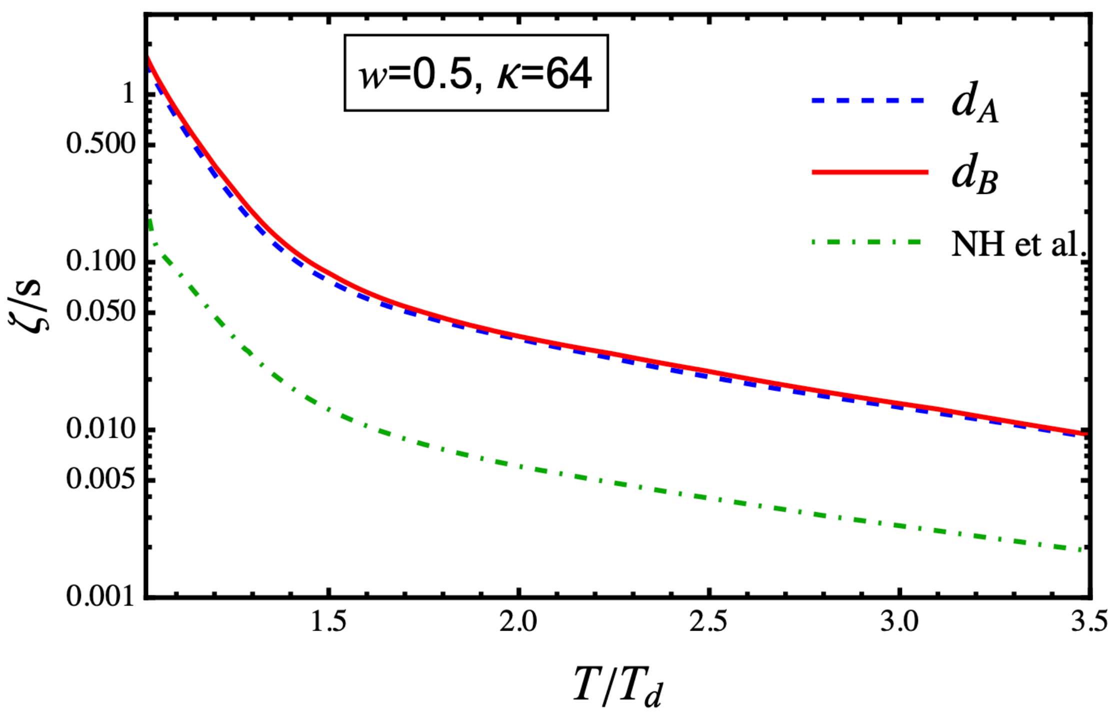


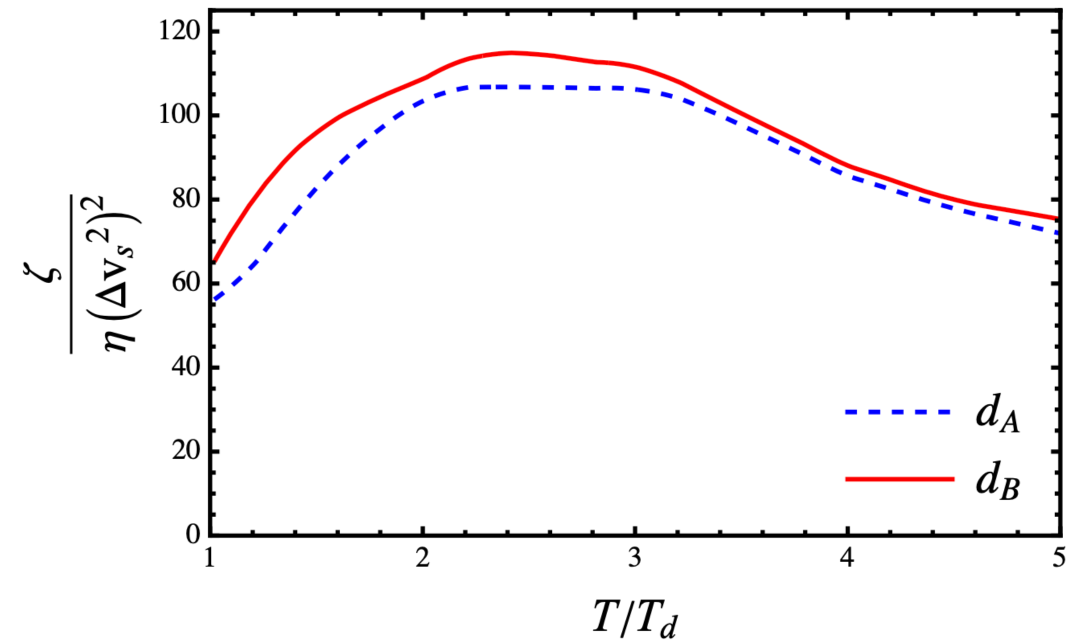
Result- Bulk Viscosity

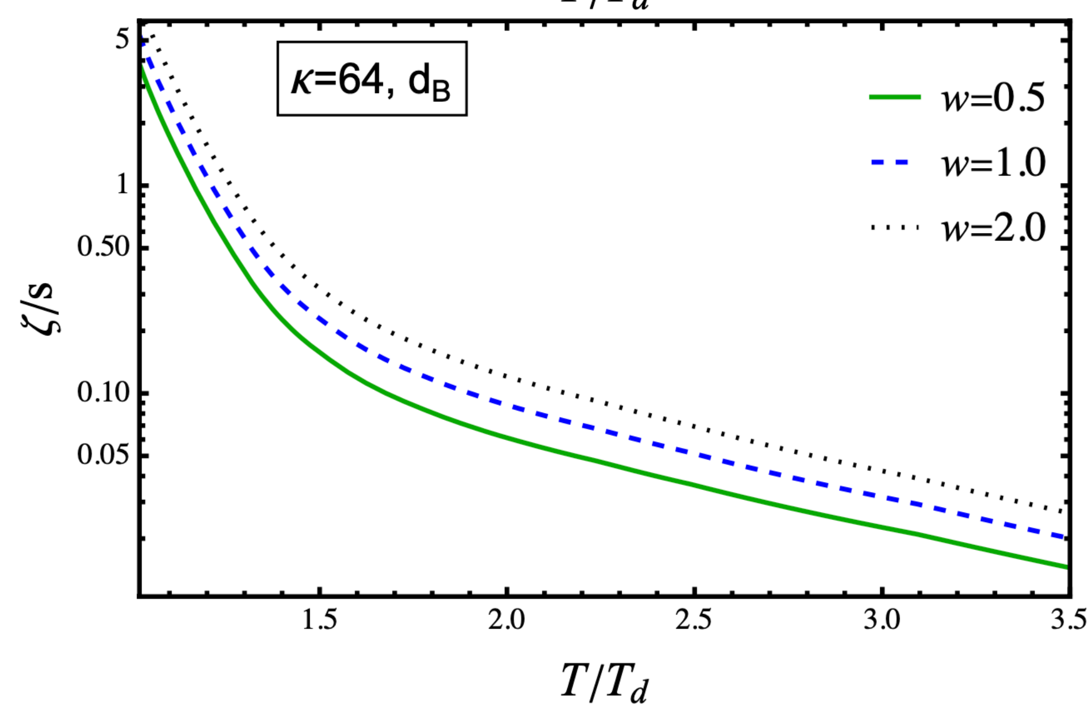
NH, R Ghosh, M Debnath, Y Hidaka, R Pisarski (in preparation)

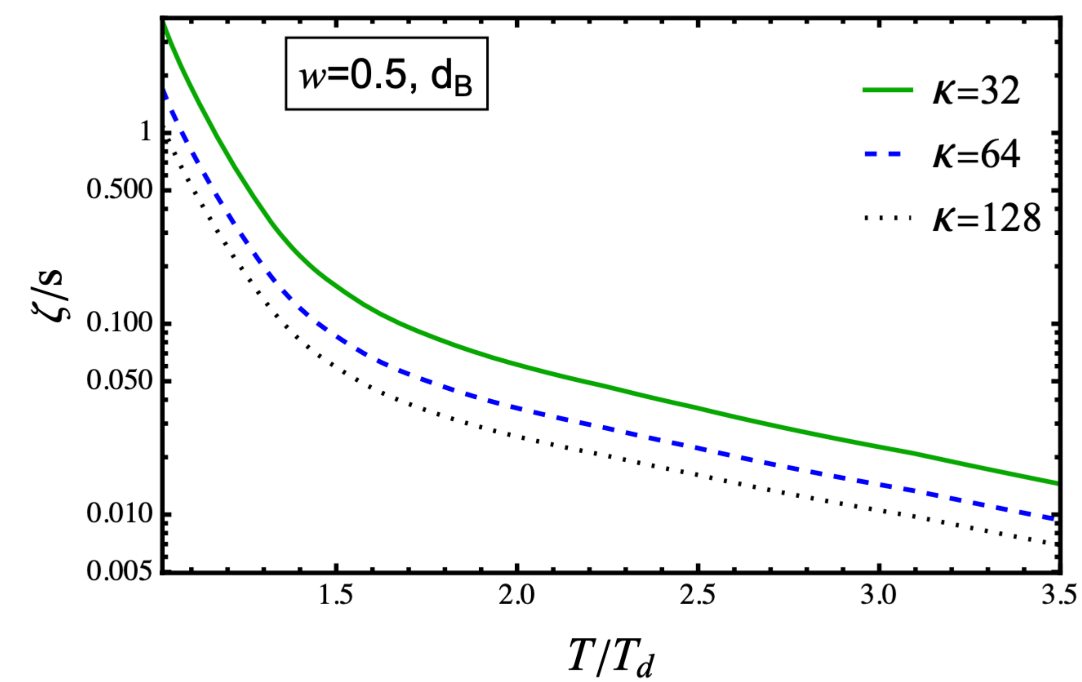
$$\zeta = \frac{T^3(\Delta v_s^2)^2}{g^4 \log(\kappa/g^2 N_c)} G(T/T_d)$$

Does not appear in perturbative study









Conclusion and outlook

- Teen field can explain the $\sim T^2$ behaviour of pressure near the transition region.

- η/s is still larger than the AdS/CFT bound even at transition temperature T_d. With Quarks, Polyakov loop value is smaller at T_d, so we expect η/s will be smaller.

- We find our bulk viscosity larger than the existing quasi-particle model result.

- We are looking forward to extend our calculation with quarks.

Thank You for your attention

Integration Measure:

For Integration over gluon momenta:

$$\int_{\mathrm{g}} d\Gamma_{ab} = \sum_{s} \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ba} \int \frac{d^3p}{(2\pi)^3 2E_p}$$

For Integration over teen(ര) momenta:

$$\int_{2\mathrm{D}} d\Gamma_{ab} = \sum_{s} \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ba} \int \frac{d^3p}{(2\pi)^3 2E_p} = \sum_{s} \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ba} \int_0^{T_d} d^2p_{\perp} \int \frac{dp_{\|} d\Omega}{(2\pi)^3 2p_{\|}}$$

$$= \sum_{s} \sum_{a,b=1}^{N_c} \mathscr{P}^{ab,ba} \frac{T_d^2}{2} \int \frac{dp_{\|} d\Omega}{(2\pi)^3 2p_{\|}}$$