In-In Effective Field Theory, The In-Out Way

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Overview

1. Introduction and Motivation
2. Theoretical Framework: In-Out vs In-In
3. Equivalence of In-In and In-Out
4. Toy Model: Explicit Calculation
5. Effective Operator Construction
6. Matching Procedure: Detailed Examples
7. Schrödinger Representation
8. Conclusions and Outlook

Introduction and Motivation

- Effective Field Theories (EFTs) provide powerful framework for separation of scales
- Traditional EFTs focus on **In-Out** correlators (S-matrix elements)
- Many physical situations require **In-In** correlators (real-time expectation values):
  - Cosmology: inflationary perturbations, CMB fluctuations
  - Non-equilibrium physics: quantum transport, thermalization
  - Condensed matter: time-dependent phenomena, quenches
  - Quantum information: decoherence, entanglement dynamics
- Key question: Is the In-In EFT same as In-Out EFT? How to systematically construct EFTs for In-In correlators?
- Taking large mass/mom. limit, low energy operators and contributions obtained. But can miss terms/effects!

Need Matching: Some examples

- *ep* scattering and taking the non-rel. limit - if working in a non-covariant gauge (common in heavy quark/electron calculations ($v \cdot A = 0$ gauge), naive NR limit leads to zero scattering ampl. Resolution: higher order terms are not suppressed! Need to take them into account
- NR limit of a real scalar field theory. Naive limit and ignoring $exp(i\, n\, mt)$ terms - fast oscillations - leads to wrong Wilson coefficients
- $b \to s\gamma$: the usual current-current operators with charm/up quarks start contributing from next order
- EFT reorganises the contributions - no one to one correspondence with the diagrams in full theory
- Matching: write operators in EFT, compute amplitudes and equate to full theory answers. Actual integrating out in PI sense doesn't work practically.

Theoretical Framework: In-Out Formalism

- In this talk: consider flat spacetime and a simple two field model (relativistic theory)

In-Out (S-matrix) formalism:

- Scattering amplitudes: $\langle \beta_{\text{Out}} | \alpha_{\text{In}} \rangle$
- Time evolution: $t = -\infty \to +\infty$
- LSZ reduction formula:

$$\langle p_1, \dots, p_n | S | k_1, \dots, k_m \rangle = \left[\prod_{i=1}^{n} \lim_{p_i^2 \to m^2} \int d^4 x_i e^{i p_i \cdot x_i} (-\partial_i^2 + m^2) \right]$$

$$\times \underbrace{\left[\prod_{j=1}^{m} \lim_{k_j^2 \to m^2} \int d^4 y_j e^{-i k_j \cdot y_j} (-\partial_j^2 + m^2) \right]}_{\text{Strips off extrnal legs}} \langle \Omega | T\{\phi(x_1) \cdots \phi(y_m)\} | \Omega \rangle$$

Theoretical Framework: In-In Formalism

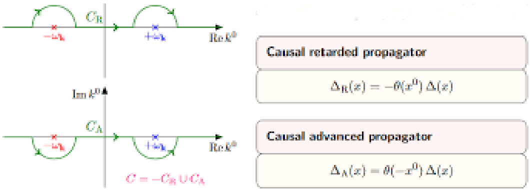
In-In (Schwinger-Keldysh) formalism:

- Expectation values: $\langle\psi(t)|\mathcal{O}(t)|\psi(t)\rangle$
- Closed time path: $t = -\infty \to t \to -\infty$
- Time translation symmetry explicitly broken
- Double field formalism: ϕ_+ (forward path), ϕ_- (backward path)
- Key quantity:

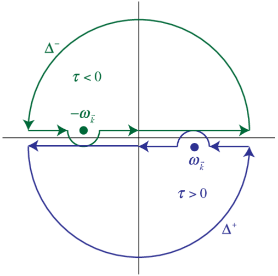
$$G_{\mathrm{In-In}}(x_1, \ldots, x_n) = \langle\Psi| T_C\{\phi(x_1)\cdots\phi(x_n)\}|\Psi\rangle$$

where T_C denotes contour ordering

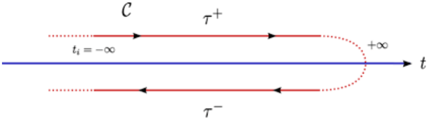
Different Contours (Images from Google)



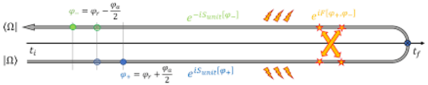
Retarded and Advanced



Feynman contour in complex plane



Closed time path contour



Closed time path contour with Operator insertions

Keldysh Rotation and Field Variables

Keldysh rotation to classical/quantum fields:

Field decomposition:

$$\varphi_- = \varphi_r + \frac{\varphi_a}{2}, \quad \varphi_+ = \varphi_r - \frac{\varphi_a}{2}$$

Generating functional:

$$Z = \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- e^{i(S[\varphi_+]-S[\varphi_-])}$$

$$\varphi_c = \frac{1}{2}(\varphi_+ + \varphi_-), \quad \varphi_q = \varphi_+ - \varphi_-$$

Action in Keldysh basis:

$$S[\varphi_c, \varphi_q] = \int dt \left[\varphi_q \frac{\delta S}{\delta \varphi_c} + \mathcal{O}(\varphi_q^3)\right]$$

Propagators in Keldysh space:

$$G^K = \frac{1}{2}\langle\{\varphi, \varphi\}\rangle, \quad G^R = i\theta(t-t')\langle[\varphi(t), \varphi(t')]\rangle$$

$$G^A = -i\theta(t'-t)\langle[\varphi(t), \varphi(t')]\rangle$$

Practical Calculations: Nested Commutators

- Path Integral way: doubling the fields - what about the vertical path? Often, one proceeds as

$$\langle \mathcal{O} \rangle(t) = \langle \psi(t)|\mathcal{O}|\psi(t)\rangle = \langle 0| \left(\overline{T} e^{+i\int_{-\infty}^{t} H_I dt'} \right) \mathcal{O} \left(T e^{-i\int_{-\infty}^{t} H_I dt'} \right) |0\rangle$$

Expand the exponentials and rearrange to have nested commutators

$$\langle \mathcal{O} \rangle(t) = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{t} dt_n \ldots \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1$$

$$\times \left\langle 0 \left[\left[\left[\ldots \left[\mathcal{O}, H_I(t_n) \right] \ldots, H_I(t_2) \right], H_I(t_1) \right] \right] 0 \right\rangle$$

- While looks simple and straightforward, intermediate steps become increasingly complicated as the number of insertions of interaction vertices increases - higher point functions and/or loops

Time Evolution Operator: Key Properties

- Interacting field: $\varphi(t,\vec{x}) = U^\dagger(t)\varphi_{\text{in}}(t,\vec{x})U(t)$
- Time evolution operator:

$$U(t) = T\left(\exp\left[-i\int_{-\infty}^{t} dt' H_{\text{int}}(t')\right]\right)$$

- Composition law: $U(t,t') = U(t)U^{-1}(t')$, $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$
- S-matrix definition: $S = \lim_{t\to\infty}\lim_{t'\to-\infty} U(t,t')$
- Important relation:

$$U(-\infty,t) = U^\dagger(t,-\infty) = \overline{\mathcal{T}}\left(\exp\left[i\int_{-\infty}^{t} dt' H_{\text{int}}(t')\right]\right)$$

Equivalence of In-In and In-Out

- Argued/shown recently by Donath and Pajer (2024)
- Rotating the contour to simplify calculations - in paper by Maldacena (2003), and several papers on cosmological correlators and connecting to Euclidean AdS (EAdS)
- Perhaps obvious but stated explicitly only recently

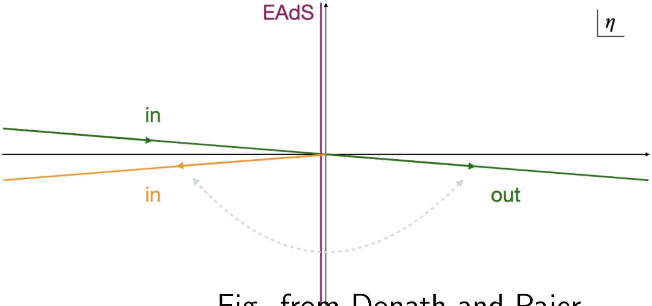


Fig. from Donath and Pajer

The Equivalence

In-In correlator definition:

$$G_{\mathrm{In-In}}(x_1,\ldots,x_n) = \langle 0|U(-\infty,t)\mathcal{O}(t)U(t,-\infty)|0\rangle$$

where $\mathcal{O}(t) = \phi(x_1)\cdots\phi(x_n)$ with all times $t_i \leq t$.
Step 1: Insert identity operator in the form $U(t,\infty)U(\infty,t) = 1$:

$$= \langle 0|U(-\infty,t)U(t,\infty)U(\infty,t)\mathcal{O}(t)U(t,-\infty)|0\rangle$$

Step 2: Combine evolution operators using composition property:

$$U(-\infty,t)U(t,\infty) = U(-\infty,\infty)$$

So we get:

$$= \langle 0|U(-\infty,\infty)U(\infty,t)\mathcal{O}(t)U(t,-\infty)|0\rangle$$

Step 3: Express in terms of in-fields using $\varphi(x) = U^\dagger(t)\varphi_{\rm in}(x)U(t)$:

$$= \langle 0|U(-\infty,\infty)T\left(\prod_i \varphi_{\rm in}(x_i)e^{-i\int_t^\infty dt' H_{\rm int}(t')}\right)U(t,-\infty)|0\rangle$$

Step 4: Note that $U(t,-\infty) = T\left(e^{-i\int_{-\infty}^t dt' H_{\rm int}(t')}\right)$

Step 5: Combine the time orderings. The product becomes:

$$T\left(\prod_i \varphi_{\rm in}(x_i)e^{-i\int_t^\infty dt' H_{\rm int}(t')}\right)U(t,-\infty) = T\left(\prod_i \varphi_{\rm in}(x_i)e^{-i\int_{-\infty}^\infty dt' H_{\rm int}}\right)$$

Time ordering organizes operators along full time contour.

Step 6: We now have:

$$G_{\mathrm{In-In}}(x_1, \ldots, x_n) = \langle 0| U(-\infty, \infty) T \left(\prod_i \varphi_{\mathrm{in}}(x_i) e^{-i \int_{-\infty}^{\infty} dt' H_{\mathrm{int}}(t')} \right) |0\rangle$$

Step 7: Express $U(-\infty, \infty)$ in terms of anti-time-ordering:

$$U(-\infty, \infty) = \overline{\mathcal{T}} \left(e^{i \int_{-\infty}^{\infty} dt H_{\mathrm{int}}(t)} \right)$$

$$\boxed{G_{\mathrm{In-In}} = \langle 0| \overline{\mathcal{T}} \left(e^{i \int_{-\infty}^{\infty} dt H_{\mathrm{int}}(t)} \right) T \left(\prod_i \varphi_{\mathrm{in}}(x_i) e^{-i \int_{-\infty}^{\infty} dt H_{\mathrm{int}}(t)} \right) |0\rangle}$$

Step 8: For vacuum expectation values, the anti-time-ordered and time-ordered parts combine:

$$\langle 0|\overline{\mathcal{T}}\left(e^{i\int_{-\infty}^{\infty}dtH_{\rm int}(t)}\right)T\left(\prod_i \varphi_{\rm in}(x_i)e^{-i\int_{-\infty}^{\infty}dtH_{\rm int}(t)}\right)|0\rangle$$

$$=\langle 0|T\left(\prod_i \varphi_{\rm in}(x_i)e^{-i\int_{-\infty}^{\infty}dtH_{\rm int}(t)}\right)|0\rangle$$

In-Out correlator definition:

$$G_{\text{In-Out}}(x_1,x_2\ldots x_n)=\langle 0|T[\varphi(x_1)\ldots\varphi(x_n)]|0\rangle$$

$$=\langle 0|U^{-1}(t)T[\prod_i \varphi_{in}(x_i)\ \exp\left[-i\int_{-t}^{t}dt' H_{int}(t')\right]]U(-t)|0\rangle$$

Recall: Time evolution operator:

$$U(t) = T\left(\exp\left[-i\int_{-\infty}^{t} dt' H_{\text{int}}(t')\right]\right)$$

Step 9: For scattering amplitudes, $t \to \infty$, and thus combining,

$$G_{\mathrm{In-Out}}(x_1, \ldots, x_n) = \langle 0|\, T\left(\prod_i \varphi_{\text{in}}(x_i) e^{-i\int_{-\infty}^{\infty} dt H_{\text{int}}(t)}\right)|0\rangle$$

Final result

$$G_{\mathrm{In-In}}(x_1, \ldots, x_n) = G_{\mathrm{In-Out}}(x_1, \ldots, x_n)$$

In-In correlation function and In-Out Green function are the same.
No amputation of the external legs - not computing S-matrix!

The usual form of In-Out correlation functions:

$$G_{\text{In-Out}}(x_1, x_2 \ldots x_n) = \langle 0 | T[\varphi(x_1) \ldots \varphi(x_n)] | 0 \rangle$$
$$= \langle 0 | U^{-1}(t) T[\prod_i \varphi_{in}(x_i) \exp \left[-i \int_{-t}^{t} dt' H_{int}(t') \right]] U(-t) | 0 \rangle$$

- For scattering amplitudes, $t \to \infty$
- action of $U(-\infty)$ on $|0\rangle$ and that of $U^{-1}(\infty)$ on $\langle 0|$ leads to the normalization factor (**leading to vacuum bubbles**)

$$(\langle 0 | S | 0 \rangle)^{-1} = (\langle 0 | T \left(\exp \left[-i \int_{-\infty}^{\infty} dt\, H_{int}(t) \right] \right) | 0 \rangle)^{-1}$$

$$\boxed{G_{\text{In-Out}}(x_1, x_2 \ldots x_n) = \frac{\langle 0 | T[\prod_i \varphi_{In}(x_i) \exp \left[-i \int_{-t}^{t} dt' H_{int}(t') \right]] | 0 \rangle}{\langle 0 | T \left(\exp \left[-i \int_{-\infty}^{\infty} dt\, H_{int}(t) \right] \right) | 0 \rangle}}$$

Practical implication of the Equivalence

- We can compute In-In correlators using standard Feynman rules!
- Would be handy and (likely to be) more efficient when considering loop effects

Toy Model Setup

Consider two scalar fields in Minkowski spacetime:

- Light field φ with $m_\varphi = 0$ (massless)
- Heavy field χ with mass M (heavy degree of freedom)
- Interaction Lagrangian: $\mathcal{L}_{\text{int}} = \frac{g}{2}\chi\varphi^2$
- Coupling constant: $g = M\kappa$ (dimensionless κ)
- Goal: Compute 4-point function $\langle\varphi\varphi\varphi\varphi\rangle$ and integrate out χ

Full theory Lagrangian:

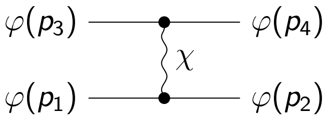
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}M^2\chi^2 - \frac{g}{2}\chi\varphi^2$$

s-Channel Calculation: Feynman Rules

Propagators:

$$\Delta_\varphi(p) = \frac{i}{p^2 + i\epsilon}, \quad \Delta_\chi(p) = \frac{i}{p^2 - M^2 + i\epsilon}$$

Vertex factor: $-ig$
s-Channel diagram:



Amplitude:

$$i\mathcal{M} = (-ig)^2 \frac{i}{(p_1 + p_2)^2 - M^2 + i\epsilon} = \frac{ig^2}{(p_1 + p_2)^2 - M^2 + i\epsilon}$$

In-Out Correlator Calculation

Full 4-point function:

$$G^{(4)}_{\mathrm{In-Out}}(p_1, p_2, p_3, p_4) = (-ig)^2 \frac{i}{(p_1+p_2)^2 - M^2 + i\epsilon} \underbrace{\prod_{i=1}^{4} \frac{i}{p_i^2 + i\epsilon}}_{Amputated}$$

Write in terms of energies and 3-momenta:

$$= (-ig)^2 \frac{i}{\omega_s^2 - E_s^2 + i\epsilon} \prod_{i=1}^{4} \frac{i}{\omega_i^2 - E_i^2 + i\epsilon}$$

where:

- $\omega_s = \omega_1 + \omega_2$ (center-of-mass energy)
- $E_s^2 = |\vec{p}_1 + \vec{p}_2|^2 + M^2$ (heavy field energy)
- $E_i = |\vec{p}_i|$ (massless particle energy)

Fourier Transform to Position Space

In-In correlator at equal time:

$$G^{(s)}_{\mathrm{In-In}}(t; |\vec{p}_i|) = \int \prod_{j=1}^{4} \frac{d\omega_j}{2\pi} e^{i\omega_j t} G^{(4)}_{\mathrm{In-Out}}(\omega_i, \vec{p}_i)$$

More explicitly:

$$G^{(s)}_{\mathrm{In-In}}(t; |\vec{p}_i|) = -ig^2 \int \prod_{j=1}^{4} \frac{d\omega_j}{2\pi} \frac{(2\pi)\delta(\sum_{j=1}^{4} \omega_j) e^{it\sum_{j=1}^{4}\omega_j}}{(\omega_{12}^2 - E_{12}^2 + i\epsilon)\prod_{i=1}^{4}(\omega_i^2 - E_i^2 + i\epsilon)}$$

where $\omega_{12} = \omega_1 + \omega_2$, $E_{12}^2 = |\vec{p}_1 + \vec{p}_2|^2 + M^2$.

Residue Calculus: Pole Structure

Poles in the complex ω-plane:

- For each ω_i: poles at $\omega_i = \pm E_i \mp i\epsilon$
- For ω_{12}: poles at $\omega_{12} = \pm E_{12} \mp i\epsilon$
- The $i\epsilon$ prescription tells us how to close contours
- We close all contours in the lower half-plane for $t > 0$

Result after residue calculus:

$$G^{(s)}_{\mathrm{In-In}}(t; |\vec{p}_i|) = -i \frac{g^2}{8E_1E_2E_3E_4E_T} \left(\frac{(E_s + E_T)}{E_s(E_s + E_{12})(E_s + E_{34})} \right)$$

where:

- $E_T = \sum_{i=1}^{4} E_i$ (total energy): Total Energy Pole - not S-matrix
- $E_{ij} = E_i + E_j$ (sum of energies)
- $E_s = \sqrt{|\vec{p}_1 + \vec{p}_2|^2 + M^2}$ (heavy field energy)

Large Mass Expansion

Expand in powers of $1/M$ using
$E_s = M\sqrt{1+\frac{p_{12}^2}{M^2}} = M + \frac{p_{12}^2}{2M} - \frac{(p_{12}^2)^2}{8M^3} + \cdots$:

$$\mathcal{O}(1/M^2) : 1$$
$$\mathcal{O}(1/M^3) : E_T - E_{12} - E_{34} = 0 \quad \text{(energy conservation)}$$
$$\mathcal{O}(1/M^4) : \frac{1}{2}\left[-p_{12}^2 - p_{34}^2 - 2(E_1+E_2)(E_3+E_4)\right]$$
$$\mathcal{O}(1/M^5) : E_T(E_1+E_2)(E_3+E_4)$$
$$\mathcal{O}(1/M^6) : \frac{1}{8}\left[(p_{12}^2)^2 + (p_{34}^2)^2 - 4(E_1+E_2)^2(E_3+E_4)^2 + \cdots\right]$$

New Feature: Odd powers of $1/M$ appear (for a rel. theory having only scalars), unlike In-Out EFT!

EFT Philosophy and Matching Condition

EFT construction principle:

$$\langle \text{In-In Correlators} \rangle_{\text{EFT}} = \langle \text{In-In Correlators} \rangle_{\text{Full Theory}}$$

Step-by-step procedure:

1. Compute correlator in full theory (with heavy fields)
2. Expand in powers of $1/M$ (heavy mass expansion)
3. Write general EFT operator basis consistent with symmetries
4. Compute correlator from each EFT operator
5. Match coefficients order by order in $1/M$

Key difference from In-Out EFT: Time translation symmetry is broken

Symmetry Considerations

Unbroken symmetries:

- Spatial translations and rotations
- $\varphi \to -\varphi$ symmetry (assuming φ is $\mathbb{Z}_2$ symmetric)
- Spatial parity

Broken symmetries:

- Time translation symmetry
- Lorentz symmetry (boosts)

Consequences:

- Time and space derivatives treated differently
- Boundary terms cannot be ignored
- Operators with odd time derivatives are allowed

Operator Classification: Spatial Derivatives

Spatial derivative operators (up to dimension 6):

- $(\partial_i \varphi)(\partial_i \varphi)\varphi^2$ (dimension 6)
- $(\partial_i \varphi^2)(\partial_i \varphi^2)$ (dimension 6)
- $\varphi^2(\partial_i^2 \varphi^2)$ (dimension 6)
- $(\partial_i^2 \varphi)(\partial_i^2 \varphi)$ (dimension 6)
- $\varphi(\partial_i^4 \varphi)$ (dimension 6)

Note: These operators are similar to In-Out EFT but with important differences in coefficients.

Operator Classification: Time Derivatives

Time derivative operators (up to dimension 7):

- $\partial_t \varphi^2$ (dimension 3) - normally total derivative
- $\partial_t \varphi^4$ (dimension 5) - normally total derivative
- $\partial_t^2 \varphi^2$ (dimension 4)
- $\partial_t^2 \varphi^4$ (dimension 6)
- $\partial_t \varphi^2 \partial_t \varphi^2$ (dimension 6)
- $\partial_t(\partial_t \varphi^2 \partial_t \varphi^2)$ (dimension 7) - boundary term
- $\partial_t^3 \varphi^2$ (dimension 5)
- $\partial_t^3 \varphi^4$ (dimension 7)

Key point: Operators that are normally total derivatives become important in In-In EFT.

Operator Classification: Mixed Derivatives

Mixed space-time derivative operators (up to dimension 7):

- $\partial_t(\varphi^2\partial_i^2\varphi^2)$ (dimension 7)
- $\partial_t\varphi^2\partial_i^2\varphi^2$ (dimension 7)
- $\partial_i^2(\varphi^2\partial_t\varphi^2)$ (dimension 7)
- $\partial_t^2(\varphi^2\partial_i^2\varphi)$ (dimension 7)
- $\partial_i^2(\partial_t\varphi)^2$ (dimension 6)

Important: Several operators break Lorentz symmetry, as expected when time translation is broken.

Effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}} =& \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}v^2(\partial_i\varphi)^2 - m^2\varphi^2 \\
&+ C_3\partial_t\varphi^2 - \frac{\lambda}{4!}\varphi^4 + C_4\partial_t^2\varphi^2 + C_5\partial_t\varphi^4 \\
&+ C_{6,1}\varphi^2(\partial_t^2\varphi^2) + C_{6,2}\partial_t^2\varphi^4 + C_{6,3}\partial_t\varphi^2\partial_t\varphi^2 \\
&+ C_{6,4}\varphi^6 + C_{6,5}(\partial_i^2)^2\varphi^2 + C_{6,6}\partial_t^4\varphi^2 \\
&+ C_{7,1}\partial_t(\varphi^2\partial_t^2\varphi^2) + C_{7,2}\partial_t\varphi^2\partial_t^2\varphi^2 + \cdots
\end{aligned}
$$

Key features:

- Different coefficients for time and space derivatives ($v^2 \neq 1$)
- Boundary terms included ($C_{7,1}$, etc.)
- Odd time derivatives allowed
- Lorentz symmetry explicitly broken

Matching Procedure: General Approach

General matching equation:

$$G^{\text{full}}_{\mathrm{In-In}}(t; \{\vec{p}_i\}) = \sum_k C_k \, G^{(k)}_{\mathrm{In-In}}(t; \{\vec{p}_i\})$$

where:

- $G^{\text{full}}_{\mathrm{In-In}}$ is the correlator computed in the full theory
- $G^{(k)}_{\mathrm{In-In}}$ is the contribution from EFT operator $\mathcal{O}_k$
- C_k are the Wilson coefficients to be determined

Strategy: Expand both sides in powers of $1/M$ and match coefficients order by order.

Matching Example 1: $\frac{\lambda}{4!}\varphi^4$ Operator

Operator: $\mathcal{O} = \frac{\lambda}{4!}\varphi^4$

Contribution to 4-point function:

$$G^{(4)}_{\mathrm{In-In}}(\lambda) = -i\lambda \int \prod_{j=1}^{4} \frac{d\omega_j}{2\pi} \frac{(2\pi)\delta(\sum_{j=1}^{4}\omega_j) e^{it\sum_{j=1}^{4}\omega_j}}{\prod_{i=1}^{4}(\omega_i^2 - E_i^2 + i\epsilon)}$$

Residue calculation:

$$= -i\lambda \frac{1}{8E_1E_2E_3E_4E_T}$$

Matching: Compare with $\mathcal{O}(1/M^2)$ term from full theory:

$$-i\lambda \frac{1}{8E_1E_2E_3E_4E_T} = -i\frac{g^2}{M^2}\frac{1}{8E_1E_2E_3E_4E_T} + \cdots$$

$$\Rightarrow \lambda = \frac{g^2}{M^2} = \kappa^2$$

Matching Example 2: $C_{7,1}\partial_t(\varphi^2\partial_t^2\varphi^2)$ Operator

Operator: $\mathcal{O} = C_{7,1}\partial_t(\varphi^2\partial_t^2\varphi^2)$ (boundary term)

Action on momentum states:

$$\partial_t(\varphi^2\partial_t^2\varphi^2) \to i(\omega_1+\omega_2+\omega_3+\omega_4)(\omega_1+\omega_2)^2$$

Contribution:

$$G^{(4)}_{\mathrm{In-In}}(C_{7,1}) = -C_{7,1}\int\prod_{j=1}^{4}\frac{d\omega_j}{2\pi}\frac{(2\pi)\delta(\sum_{j=1}^4\omega_j)e^{it\sum_{j=1}^4\omega_j}}{\prod_{i=1}^4(\omega_i^2-E_i^2+i\epsilon)}(\omega_1+\omega_2)^3$$

Residue calculation:

$$= C_{7,1}\frac{(E_1+E_2)(E_3+E_4)E_T}{8E_1E_2E_3E_4E_T}$$

Matching: Compare with $\mathcal{O}(1/M^5)$ term from full theory:

$$C_{7,1} = -i\frac{g^2}{M^5} = -i\frac{\kappa^2}{M^3}$$

In-In vs In-Out EFT

EFT construction principle:

$$\langle \text{Physical Quantity} \rangle_{\text{EFT}} = \langle \text{Physical Quantity} \rangle_{\text{Full Theory}}$$

Differences between the two EFTs:

1. Amplitudes (amputated Green functions) vs Non-amputated, evaluated at a fixed time
2. In-In: total derivatives in time - crucial, and in general will lead to different coefficients between otherwise Lorentz invariant combinations
3. In both cases, write general EFT operator basis consistent with symmetries of each and match
4. Compute correlator from each EFT operator

But now, both computed with same steps and manipulations - the usual Feynman way

Gaining more confidence - Schrödinger Picture

Key ideas: like Quantum Mechanics

- Field eigenstates: $\varphi(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle$
- Wavefunctional: $\Psi[\phi] = \langle\phi|\Psi\rangle$
- Conjugate momentum: $\pi(\vec{x}) = -i\frac{\delta}{\delta\phi(\vec{x})}$
- Canonical commutation: $[\varphi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

Schrödinger equation:

$$i\frac{\partial}{\partial t}\Psi[\phi, t] = H\Psi[\phi, t]$$

where

$$H = \frac{1}{2}\int d^3x\left(-\frac{\delta^2}{\delta\phi^2(\vec{x})} + |\nabla\phi|^2 + m^2\phi^2\right) + H_{\rm int}$$

Free Theory Solution

Free Hamiltonian:

$$H_0 = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\vec{x})} + |\nabla\phi|^2 + m^2\phi^2\right)$$

Ground state wavefunctional:

$$\Psi_0[\tilde{\phi}] = \prod_{\vec{k}} \left(\frac{\omega_k}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}\frac{1}{(2\pi)^3}\omega_k \tilde{\phi}^2(|\vec{k}|)\right)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$, $\tilde{\phi}$ is Fourier transform.
Verification:

$$H_0\Psi_0[\phi] = E_0\Psi_0[\phi], \quad E_0 = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega_k$$

Creation and Annihilation Operators

In terms of field and functional derivative:

$$a(\vec{k}) = \int d^3x e^{i\vec{k}\cdot\vec{x}} \left(\omega_k \phi(\vec{x}) + \frac{\delta}{\delta\phi(\vec{x})} \right)$$

$$a^\dagger(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left(\omega_k \phi(\vec{x}) - \frac{\delta}{\delta\phi(\vec{x})} \right)$$

Verify commutation relations:

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

Action on vacuum:

$$a(\vec{k})\Psi_0[\phi] = 0, \quad a^\dagger(\vec{k})\Psi_0[\phi] = \sqrt{(2\pi)^3 2\omega_k}\Psi_1[\phi; \vec{k}]$$

ϕ^4 Theory: Perturbation Setup

Full Hamiltonian: $H = H_0 + H_{\text{int}}$ with

$$H_{\text{int}} = \frac{\lambda}{4!} \int d^3x \phi^4(\vec{x})$$

Perturbation theory:

$$\Psi_n[\phi] = \Psi_n^{(0)}[\phi] + \lambda \Psi_n^{(1)}[\phi] + \lambda^2 \Psi_n^{(2)}[\phi] + \cdots$$
$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$$

First order correction:

$$\Psi_0^{(1)}[\phi] = -\sum_{m \neq 0} \frac{\langle m | H_{\text{int}} | 0 \rangle}{E_m^{(0)} - E_0^{(0)}} \Psi_m^{(0)}[\phi]$$

S-Matrix Calculation in Schrödinger Picture

S-matrix element:

$$S_{\beta\alpha} = \int \mathcal{D}\phi \, \Psi^*_{p_3 p_4}[\phi] \Psi_{p_1 p_2}[\phi]$$

Expand to first order in λ:

$$\begin{aligned} S_{\beta\alpha} &= \int \mathcal{D}\phi \left(\Psi_0^{*(0)} + \lambda \Psi_0^{*(1)} + \cdots \right) a(\vec{p}_3) a(\vec{p}_4) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) \\ &\qquad \otimes \left(\Psi_0^{(0)} + \lambda \Psi_0^{(1)} + \cdots \right) \\ &= S^{(0)}_{\beta\alpha} + \lambda S^{(1)}_{\beta\alpha} + \cdots \end{aligned}$$

Detailed First Order Calculation

Zeroth order: $S^{(0)}_{\beta\alpha} = 0$ (no tree-level ϕ^4 vertex)
First order:

$$S^{(1)}_{\beta\alpha} = \int \mathcal{D}\phi\, \Psi^{*(1)}_0[\phi] a(\vec{p}_3) a(\vec{p}_4) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) \Psi^{(0)}_0[\phi] + \int \mathcal{D}\phi\, \Psi^{*(0)}_0[\phi] a(\vec{p}_3) a(\vec{p}_4) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) \Psi^{(1)}_0[\phi]$$

After explicit calculation:

$$S^{(1)}_{\beta\alpha} \sim -\lambda (2\pi)^3 \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4) \frac{e^{-it(E_{p_1} + E_{p_2})}}{E_{p_1} + E_{p_2} + E_{p_3} + E_{p_4}}$$

Total Energy Pole Structure

Key result:

$$S^{(1)}_{\beta\alpha} \sim -\lambda(2\pi)^3\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)\frac{e^{-it(E_{p_1}+E_{p_2})}}{E_{p_1} + E_{p_2} + E_{p_3} + E_{p_4}}$$

Features:

- Total energy pole: $1/(E_{p_1} + E_{p_2} + E_{p_3} + E_{p_4})$
- Time dependence: $e^{-it(E_{p_1}+E_{p_2})}$
- 3-momentum conservation: $\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$

Note: external propagators amputated/stripped-off as S-matrix was being computed.

Connection to In-In formalism: At finite t, this has the structure of In-In correlators with the characteristic total energy pole.

Taking the $t \to \infty$ Limit

To recover S-matrix, take $t \to \infty$ using:

$$\lim_{T\to\infty} \frac{e^{iET}}{E} = 2\pi i \delta(E)$$

Recover the expected Result:

$$S^{(1)}_{\beta\alpha} = -\lambda(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$

Observation: The Schrödinger picture naturally contains both the finite-time structure (In-In) and the asymptotic limit (In-Out).

Broader Amplitude Programme

- Amplituhedron, Cosmohedra, Polytopes.....: rich programme in the last one decade or so
- Aim is to have a deeper and possibly geometric understanding of the differet amplitudes with a focus on cosmological correlators
- Main object of interest: the wavefunction which is expanded in wavefunction coefficients
- Structure is very similar to the Wavefunctional in Schrodinger picture above
- We noticed that Schrödinger representation naturally captures In-In structure
- Connection with Kosower, Maybee and O'Connell (KMOC) - classical limit of the amplitudes for radiated energy, waveforms... - work in progress. In-Out methods seem to be working

Comparison with earlier works

- Several papers have computed cosmological correlators using Schwinger-Keldysh (PI or nested commutators). They then take large mass limit and write some operators.
- In recent times, following the wavefunction approach, Pajer and co. (arXiv:2212.08009) and Green and co. (arXiv: 2412.02739) have written down EFT for In-In correlation functions.
- They use equations of motion (now including the homogeneous parts), have to do a lot of manipulations to handle the surface terms and then identify effective operators

Present approach: relies on symmetries and EFT constructions, avoids intermediate cumbersome steps and also uses the more familar Feynman way of calculating

Disentangling different scales is easier this way

Doesn't commit to a particular UV model - let the experimental data pick the type of operators

Summary

- **Established equivalence**:
  $G_{\mathrm{In-In}}(x_1, \ldots, x_n) = G_{\mathrm{In-Out}}(x_1, \ldots, x_n)$
- **Time translation breaking**: Fundamental difference from In-Out EFT
- **Boundary terms**: Cannot be ignored in In-In formalism
- **Lorentz symmetry**: Explicitly broken in low-energy EFT
- **Operator basis**: Rich structure with time and space derivatives separated
- **Schrödinger picture**: Provides natural framework for understanding In-In/In-Out connection

Future Research Directions

- **Loop-level calculations**: Test equivalence beyond tree level
- **Renormalization group**: Wilson coefficient evolution in In-In EFT - mixing of In-In and In-Out effects
- **Cosmological applications**: Primordial non-Gaussianities
- **Non-equilibrium physics**: Thermalization, quantum quenches
- **Schrödinger picture**: Develop computational techniques for wavefunctionals
- **General states**: Extend beyond vacuum states

Thank you!

Comments/Suggestions?

Backup Material

ep Scattering, NR Limit and Choice of Gauge

- Leading amplitude:

$$\mathcal{M} \sim \bar{u}_P(p+k)\gamma_\mu u_P(p) D^{\mu\nu}(k) \bar{u}_e(q-k)\gamma_\nu u_e(q)$$

- Gordon decomposition:

$$\bar{u}_P(p+k)\gamma^\mu u_P(p) = \bar{u}_P(p+k)\left(\frac{(2p+k)^\mu}{2m_P} + i\frac{\sigma^{\mu\nu}k_\nu}{2m_P}\right) u_P(p)$$

- Covariant gauge propagator:

$$D^{\mu\nu}(k) = \frac{\eta^{\mu\nu} - k^\mu k^\nu / k^2}{k^2}$$

- Using on-shell condition: $p^2 = p'^2 = m_P^2$

$$2m_P k_0 + k_0^2 - \vec{k}^2 = 0 \quad \Rightarrow \quad k_0 \approx \frac{\vec{k}^2}{2m_P}$$

- Heavy proton: $k/m_P \to 0$, Gordon decomposition yields:

$$\bar{u}_P(p+k)\gamma^\mu u_P(p) \to \bar{u}_P(p+k)v^\mu u_P(p)$$

- As $k_0 \to 0$, amplitude reduces to:

$$\frac{2m_P}{-\vec{k}^2}\bar{u}_e(q-k)\gamma^0 u_e(q) + \mathcal{O}(1/m_P)$$

- Reproduces Coulomb scattering in external field:

$$A_\mu(x) = (ie/4\pi r, \vec{0})$$

Gauge Invariance and Subtleties

- Physics is gauge invariant. If instead working in different gauge:
- In gauge $v \cdot A = 0$, propagator becomes:

$$D_{\mu\nu}(k, v) = \left(\eta_{\mu\nu} - \frac{v_\mu k_\nu + v_\nu k_\mu}{v \cdot k} + \frac{k_\mu k_\nu}{(v \cdot k)^2}\right)\frac{1}{k^2}$$

- But $v^\mu D_{\mu\nu}(k, v) = 0$ - implying no scattering!

Important Step

$$k_\mu D^{\mu\nu}(k, v) = \frac{k^\nu - v^\nu v \cdot k}{(v \cdot k)^2} \sim -\frac{v^\nu}{v \cdot k} \approx -\frac{2m_P v^\nu}{\vec{k}^2}$$

Although $k_\mu / 2m_P$ is $\mathcal{O}(1/m_P)$, it cannot be neglected due to enhancement from propagator.

Resolution: Non-covariant gauges mix terms of different orders in $1/m_P$ - can be avoided by writing EFT operators and matching

Relativistic Starting Point

Lagrangian Density

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

Hamiltonian Density

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$

Equations of Motion

$$\dot{\phi} = \pi$$

$$\dot{\pi} = (\nabla^2 - m^2)\phi - \frac{1}{3!}\lambda\phi^3$$

Field Redefinition Strategy

- Standard approach (local):

$$\phi(t,\mathbf{x}) = \frac{1}{\sqrt{2m}}\left[e^{-imt}\psi(t,\mathbf{x}) + e^{imt}\psi^*(t,\mathbf{x})\right]$$

$$\pi(t,\mathbf{x}) = -i\sqrt{\frac{m}{2}}\left[e^{-imt}\psi(t,\mathbf{x}) - e^{imt}\psi^*(t,\mathbf{x})\right]$$

- New approach (nonlocal):

$$\phi(t,\mathbf{x}) = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2}\left[e^{-imt}\psi(t,\mathbf{x}) + e^{imt}\psi^*(t,\mathbf{x})\right]$$

$$\pi(t,\mathbf{x}) = -i\sqrt{\frac{m}{2}}\mathcal{P}^{1/2}\left[e^{-imt}\psi(t,\mathbf{x}) - e^{imt}\psi^*(t,\mathbf{x})\right]$$

where $\mathcal{P} \equiv \sqrt{1 - \frac{\nabla^2}{m^2}}$

Advantages of Nonlocal Redefinition

- Free theory preserves manifest $U(1)$ symmetry
- No fast-oscillating terms in free limit
- Simplified commutation relations:

$$[\psi(t, \mathbf{x}), \psi^*(t, \mathbf{y})] = \delta^3(\mathbf{x} - \mathbf{y})$$

- Exact inverse transformation:

$$\psi(t, \mathbf{x}) = \sqrt{\frac{m}{2}} e^{imt} \mathcal{P}^{1/2} \left(\phi(t, \mathbf{x}) + \frac{i}{m} \mathcal{P}^{-1} \pi(t, \mathbf{x}) \right)$$

Equation of Motion for ψ

Exact Equation

$$i\dot{\psi} = m(\mathcal{P} - 1)\psi + \frac{\lambda e^{imt}}{4!m^2}\mathcal{P}^{-1/2}\left[e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^*\right]^3$$

Key Calculation Step

$$\begin{aligned}
\dot{\phi} + \frac{i}{m}\mathcal{P}^{-1}\dot{\pi} &= \pi + \frac{i}{m}\mathcal{P}^{-1}\left[(\nabla^2 - m^2)\phi - \frac{1}{3!}\lambda\phi^3\right] \\
&= \pi - im\mathcal{P}\phi - \frac{i\lambda}{3!m}\mathcal{P}^{-1}\phi^3 \\
&= -im\mathcal{P}\left[\phi + \frac{i}{m}\mathcal{P}^{-1}\pi\right] - \frac{i\lambda}{3!m}\mathcal{P}^{-1}\phi^3
\end{aligned}$$

Lagrangian Formulation

Nonrelativistic Lagrangian

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}\psi^* - \psi\dot{\psi}^*) - m\psi^*(\mathcal{P}-1)\psi - \frac{\lambda}{4\cdot 4!m^2}\left[e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi\right.$$

Canonical Variables

$$\psi_c(t,\mathbf{x}) \equiv \sqrt{2}\psi_R(t,\mathbf{x})$$
$$\pi_c(t,\mathbf{x}) \equiv \sqrt{2}\psi_I(t,\mathbf{x})$$
$$\psi(t,\mathbf{x}) = \frac{1}{\sqrt{2}}(\psi_c(t,\mathbf{x}) + i\pi_c(t,\mathbf{x}))$$

Hamiltonian Formulation

Hamiltonian Density

$$\mathcal{H} = m\psi^*(\mathcal{P} - 1)\psi + \frac{\lambda}{4 \cdot 4! m^2} \left[e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^* \right]^4$$

Canonical Commutation

$$[\psi_c(t, \mathbf{x}), \pi_c(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

- Transformation from ϕ to ψ is canonical
- Preserves Poisson bracket/commutation relations

Mode Decomposition

Harmonic Expansion

$$\psi(t, \mathbf{x}) = \sum_{\nu=-\infty}^{\infty} \psi_\nu(t, \mathbf{x}) e^{i\nu mt}$$

where $\psi_\nu(t, \mathbf{x})$ are slowly varying

Slow Mode Definition

$$\psi_{\nu=0}(t, \mathbf{x}) \equiv \psi_s(t, \mathbf{x})$$

Equation for Mode ν

$$i\dot{\psi}_\nu - \nu m \psi_\nu = m(\mathcal{P} - 1)\psi_\nu + \frac{\lambda}{4! m^2} \tilde{G}_\nu$$

Perturbative Framework

- Small parameters:

$$\frac{\nabla^2 F}{m^2} \sim \epsilon_x F$$
$$\frac{\dot{F}}{m} \sim \epsilon_t F$$
$$\lambda \ll 1$$

- Iterative solution:

$$\Psi_\nu = -\frac{i}{m}\Gamma_\nu \dot{\Psi}_\nu + \lambda G_\nu$$

where $\Gamma_\nu \equiv (1 - \nu - \mathcal{P})^{-1}$, $\Psi_\nu \equiv \mathcal{P}^{-1/2}\psi_\nu$

- Zeroth order:

$$\Psi_\nu^{(0)} = \begin{cases} \Psi_s & \text{if } \nu = 0 \\ 0 & \text{if } \nu \neq 0 \end{cases}$$

First Order Calculation

First Order Correction

$$\Psi_\nu^{(1)} = \lambda G_\nu^{(0)} \quad \text{for } \nu \neq 0$$

Source Term

$$G_\nu^{(0)} = \frac{\Gamma_\nu \mathcal{P}^{-1}}{4! m^3} \left\{ \Psi_s^3 \delta_{\nu,-2} + \Psi_s^{*3} \delta_{\nu,4} + 3|\Psi_s|^2 \Psi_s^* \delta_{\nu,2} + 3|\Psi_s|^2 \Psi_s \delta_{\nu,0} \right\}$$

Effective Equation for ψ_s

$$i\dot{\psi}_s = m(\mathcal{P} - 1)\psi_s + m\lambda \Gamma_0^{-1} \mathcal{P}^{1/2} G_0$$

Low-Energy Effective Theory

Expanded Equation of Motion

$$
\begin{aligned}
i\dot{\psi}_s \simeq & -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s \\
& - \frac{1}{8m^3}\nabla^4\psi_s + \frac{\lambda}{32m^4}\left[\psi_s^2\nabla^2\psi_s^* + 2|\psi_s|^2\nabla^2\psi_s + \nabla^2(|\psi_s|^2\psi_s)\right] \\
& - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

Corresponding Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} = & \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2m}\nabla\psi_s\nabla\psi_s^* - \frac{\lambda}{16m^2}|\psi_s|^4 \\
& + \frac{1}{8m^3}\nabla^2\psi_s\nabla^2\psi_s^* - \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\nabla^2\psi_s + \psi_s\nabla^2\psi_s^*) \\
& \quad \frac{17\lambda^2}{}
\end{aligned}
$$

Key Features of Effective Theory

- Global $U(1)$ symmetry preserved to all orders
- Particle number conserved
- Single relativistic interaction $\rightarrow$ multiple nonrelativistic interactions
- Relativistic corrections comparable to interaction corrections:

$$\frac{\lambda\mathcal{N}}{m^3 v^2} \sim \mathcal{O}(1)$$

- Higher-order time derivatives can be systematically removed

Comparison with Previous Work

- Matches Braaten et al. (2016) for λ^2 term
- Includes additional relativistic corrections (∇^4, $\lambda\nabla^2$)
- Equivalent to Mukaida et al. (2016) after field redefinition
- Resolves apparent discrepancies through proper treatment of:
  - Canonical transformations
  - Higher-order time derivatives
  - Field redefinitions

Matching Example 3: $C_{6,1}\varphi^2(\partial_i^2\varphi^2)$ Operator

Operator: $\mathcal{O} = C_{6,1}\varphi^2(\partial_i^2\varphi^2)$
Action on momentum states: $\partial_i^2 \to -p_{ij}^2$ where $p_{ij}^2 = |\vec{p}_i + \vec{p}_j|^2$
Contribution:

$$G^{(4)}_{\mathrm{In-In}}(C_{6,1}) = -iC_{6,1}\int \prod_{j=1}^{4} \frac{d\omega_j}{2\pi} \frac{(2\pi)\delta(\sum_{j=1}^4 \omega_j) e^{it\sum_{j=1}^4 \omega_j}}{\prod_{i=1}^4(\omega_i^2 - E_i^2 + i\epsilon)}$$

$$(p_{12}^2 + p_{34}^2 + \text{permutations})$$

Residue calculation:

$$= iC_{6,1}\frac{(p_{12}^2 + p_{34}^2 + \text{permutations})}{8E_1E_2E_3E_4E_T}$$

Matching: Compare with $\mathcal{O}(1/M^4)$ term from full theory to determine $C_{6,1}$.

Matching Example 4: $C_{6,3}\partial_t\varphi^2\partial_t\varphi^2$ Operator

Operator: $\mathcal{O} = C_{6,3}\partial_t\varphi^2\partial_t\varphi^2$

Action on momentum states: $\partial_t \to i(\omega_i + \omega_j)$

Contribution:

$$G^{(4)}_{\mathrm{In-In}}(C_{6,3}) = -iC_{6,3}\int\prod_{j=1}^{4}\frac{d\omega_j}{2\pi}\frac{(2\pi)\delta(\sum_{j=1}^{4}\omega_j)e^{it\sum_{j=1}^{4}\omega_j}}{\prod_{i=1}^{4}(\omega_i^2 - E_i^2 + i\epsilon)}(\omega_1+\omega_2)(\omega_3+\omega$$

Use energy conservation: $\omega_1 + \omega_2 = -(\omega_3 + \omega_4)$

$$= iC_{6,3}\int\prod_{j=1}^{4}\frac{d\omega_j}{2\pi}\frac{(2\pi)\delta(\sum_{j=1}^{4}\omega_j)e^{it\sum_{j=1}^{4}\omega_j}}{\prod_{i=1}^{4}(\omega_i^2 - E_i^2 + i\epsilon)}(\omega_1+\omega_2)^2$$

Residue calculation:

$$= iC_{6,3}\frac{(E_1+E_2)(E_3+E_4)}{8E_1E_2E_3E_4E_T}$$

Matching: Compare with $\mathcal{O}(1/M^4)$ term from full theory.

Wavefunction of the Universe

- Correlations generated by probability distribution: $|\Psi|^2$
- Spatial correlations $\Leftrightarrow$ Wavefunction of the universe
- Wavefunction provides density distribution for spatial averages

Transition Amplitude and Wavefunction

$$\langle \Phi | U | \Phi' \rangle = \langle \Phi | \hat{\mathcal{T}} \{ e^{-i \int_{-T}^{0} dt H(t)} \} | \Phi' \rangle$$
$$= \mathcal{N} \int_{\phi(-T)=\Phi'}^{\phi(0)=\Phi} \mathcal{D}\phi e^{iS[\phi]}$$

Taking $T \to \infty(1 - i\varepsilon)$ gives vacuum wavefunction:

$$\mathcal{N} \int_{\phi(-\infty(1-i\varepsilon))=0}^{\phi(0)=\Phi} \mathcal{D}\phi e^{iS[\phi]} = \Psi_\circ[\Phi]$$

Properties of the Wavefunction

- Depends on configuration Φ at fixed time slice $t = 0$
- Time evolution completely integrated out
- Time translation invariance broken
- Lorentz invariance reduced to spatial rotations
- Energy not well-defined
- Boundary conditions select Bunch-Davies state (positive frequency)

Simple Example: Harmonic Oscillator

Action with time-dependent coefficients:

$$S[\phi] = -\int_{-\infty}^{0} d\eta \left[\frac{1}{2}\kappa(\eta)\dot{\phi}^2 - \frac{1}{2}\omega^2(\eta)\phi^2 \right]$$

Vacuum wavefunction:

$$\Psi_\circ[\Phi] = \mathcal{N} e^{iS_{cl}[\Phi]}$$

Classical solution:

$$\phi_{cl}(\eta) = \Phi \frac{\phi_+(\eta)}{\phi_+(0)}, \qquad \lim_{\eta \to -\infty(1-i\varepsilon)} \phi_+(\eta) \sim e^{i\tilde{\omega}\eta} = 0$$

Perturbative Wavefunction

Split action: $S[\phi] = S_2[\phi] + S_{\text{int}}[\phi]$ Split field:

$\phi(\eta, \vec{x}) = \phi_\circ(\eta, \vec{x}) + \varphi(\eta, \vec{x})$ Boundary conditions:

$$\begin{aligned} \phi_\circ(-\infty(1-i\varepsilon), \vec{x}) &= 0, & \phi_\circ(0, \vec{x}) &= \Phi(\vec{x}) \\ \varphi(-\infty(1-i\varepsilon), \vec{x}) &= 0, & \varphi(0, \vec{x}) &= 0 \end{aligned}$$

Path Integral Expression

$$\Psi_\circ[\Phi] = \mathcal{N} e^{iS_2[\Phi]} \int_{\varphi(-\infty(1-i\varepsilon))=0}^{\varphi(0)=0} \mathcal{D}\varphi e^{iS_2[\varphi]+iS_{\text{int}}[\Phi,\varphi]}$$

Perturbative expansion:

$$\Psi_\circ[\Phi] = e^{iS_2[\Phi]} \sum_{j=0}^{\infty} \frac{i^j}{j!} \langle S_{\text{int}}^j \rangle [\Phi]$$

Feynman Rules

- Interaction term:

$$S_{\text{int}}[\phi_\circ, \varphi] = \int d^d x \int_{-\infty}^{0} d\eta \lambda_k(\eta) V_k(\phi_\circ, \varphi, \partial_\eta, \partial_i)$$

- Bulk-to-bulk propagator $G(y_e; \eta_{v_e}, \eta_{v'_e})$ with three terms
- Wavefunction expansion:

$$\Psi_\circ[\Phi] = e^{iS_2[\Phi]} \left\{ 1 + \sum_{k\geq 2} \int \prod_{j=1}^{k} \left[\frac{d^d p^{(j)}}{(2\pi)^d} \Phi(\vec{p}^{(j)}) \right] \sum_{L=0}^{\infty} \psi_k^{(L)} \right\}$$

Probability Distribution

$$|\Psi_\circ[\Phi]|^2 = e^{-2\mathrm{Im}(S_2[\Phi])}\left\{1 + \text{interaction terms}\right\}$$

Expectation values:

$$\langle \mathcal{O}[\Phi] \rangle = \mathcal{N} \int \mathcal{D}\Phi |\Psi_\circ[\Phi]|^2 \mathcal{O}[\Phi]$$

Correlation Functions

$$\left\langle \prod_{j=1}^{n} \Phi(\vec{q}^{(j)}) \right\rangle = \mathcal{N} \int \mathcal{D}\Phi |\Psi_\circ(\Phi)|^2 \prod_{j=1}^{n} \Phi(\vec{q}^{(j)})$$

Connected contributions from wavefunction coefficients:

$$\left\langle \prod_{j=1}^{n} \Phi(\vec{q}^{(j)}) \right\rangle \sim \mathcal{N} \int \mathcal{D}\Phi \prod_{j=1}^{n} \Phi(\vec{q}^{(j)}) e^{-2\mathrm{Im}\{S_2[\Phi]\}} \sum \text{wavefunction term}$$