#### Asian School on Lattice Field Theory

# Prepotentials for SU(3) Lattice Gauge Theory

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**Outline of The Talk:** 

#### **Motivations**

Prepotential Formulation of SU(2) LGT

Generalizing to SU(3):
The Difficulties

SU(3) Prepotentials

#### **Motivation:**

Gauge Theory
Kinematical
Hilbert Space
Containing

Redundant

(Unphysical) States

**Physical Hilbert Space** 

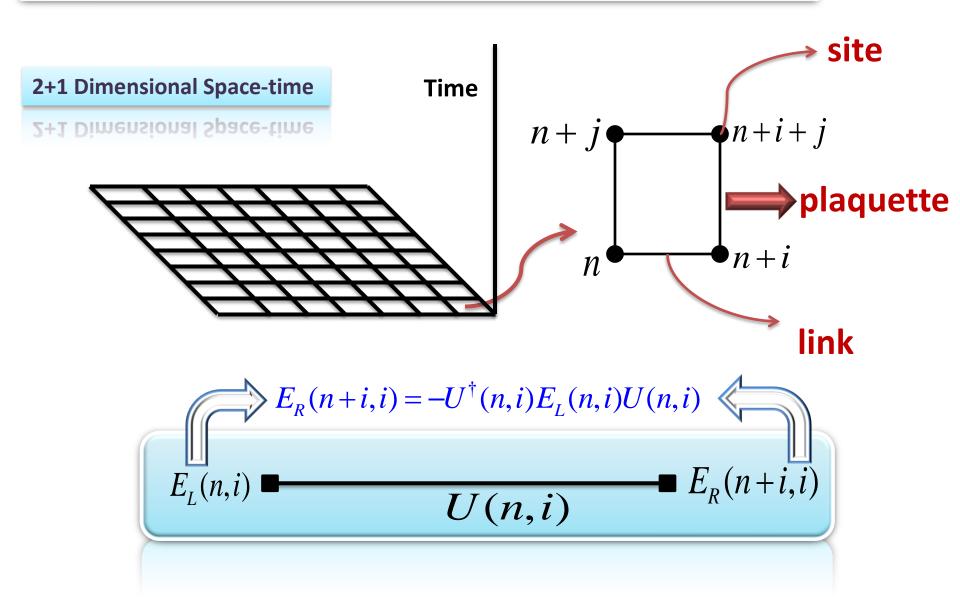
**Containing Gauge Invariant States** 

Mandelstam Constraints

**Over Complete Basis** 

Aim is to find ALL the Mandelstam Constraints for SU(3) Gauge theory on a lattice

# Lattice Gauge Theory: Hamiltonian Formulation



# Hamiltonian formulation of LGT

$$H = \sum_{n,i} \sum_{a=1}^{N^2 - 1} E^2(n,i) + K \sum_{plaquette} \operatorname{Tr}(U_{plaquette} + U^{\dagger}_{plaquette})$$

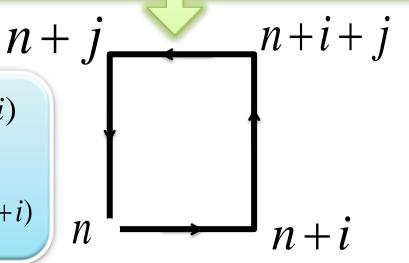
$$U_{plaquette} = U(n,i)U(n+i,j)U^{\dagger}(n+j,i)U^{\dagger}(n,j)$$

**Under Gauge Transformation:** 

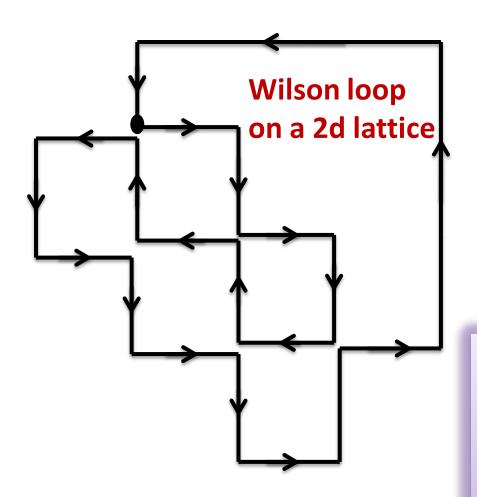
$$U(n,i) \to \Lambda(n)U(n,i)\Lambda^{\dagger}(n+i)$$

$$E_L(n,i) \to \Lambda(n)E_L(n,i)\Lambda^{\dagger}(n)$$

$$E_R(n+i,i) \to \Lambda(n+i)E_R(n+i,i)\Lambda^{\dagger}(n+i)$$



#### Gauge Invariant Objects: Wilson loops



Drawbacks of Conventional Variables

- •Infinite Number Of Loops Are Possible.
- Over Complete Basis.
- •Dynamical Variables are Highly Nonlocal.

Therefore, it is important to explore new descriptions of QCD where the loop, string states and their dynamics as well as the associated Mandelstam constraints can be analyzed locally.



# The Prepotential Approach To Lattice Gauge Theories Provides Such A Platform.

More precisely, this approach allows us to analyze and solve the Mandelstam constraints locally at each lattice site without all the irrelevant non-local details associated with the loop states. Towards this goal, a complete analysis was carried out for SU(2) lattice gauge theory and all mutually independent loop states were constructed in terms of prepotential operators.

Manu Mathur, Nucl. Phys. B 779, 32 (2007)

# Present Goal: Formulate SU(3) LGT in Prepotential Framework

#### New Variables: Prepotentials

#### SU(2) Case:

$$e^{i\theta}a_{\alpha}^{\dagger}(L) = a^{\dagger}(L) \bullet a(L) = a^{\dagger}(R) \bullet a(R)$$

$$a_{\alpha}^{\dagger}(L) \bullet a(L) = a^{\dagger}(R) \bullet a(R)$$

$$n \bullet \bullet n + i$$

$$E_{L}^{a}(n,i) \qquad E_{R}^{a}(n+i,i)$$

$$a^{\dagger}(n,i;L) \frac{\sigma^{a}}{2} a(n,i;L) \qquad a^{\dagger}(n+i,i;R) \frac{\sigma^{a}}{2} a(n+i,i;R)$$

$$U = \underbrace{\begin{pmatrix} a_{2}^{\dagger}(L)\eta_{L} & a_{1}(L)\theta_{L} \\ -a_{1}^{\dagger}(L)\eta_{L} & a_{2}(L)\theta_{L} \end{pmatrix}}_{U_{L}} \underbrace{\begin{pmatrix} \eta_{R}a_{1}^{\dagger}(R) & \eta_{R}a_{2}^{\dagger}(R) \\ \theta_{R}a_{2}(R) & \theta_{R}(-a_{1}(R)) \end{pmatrix}}_{U_{R}}$$

#### **Under Gauge Transformation:**

$$a_{\alpha}^{\dagger}(L) \rightarrow a_{\beta}^{\dagger}(L) \left(\Lambda_{L}^{\dagger}\right)^{\beta}_{\alpha} \quad a_{\alpha}^{\dagger}(R) \rightarrow a_{\beta}^{\dagger}(R) \left(\Lambda_{R}^{\dagger}\right)^{\beta}_{\alpha}$$

#### Constraints

$$G(n) = \sum_{i=1}^{d} (E_L^{a}(n,i) + E_R^{a}(n,i)) = 0 \quad \forall n$$

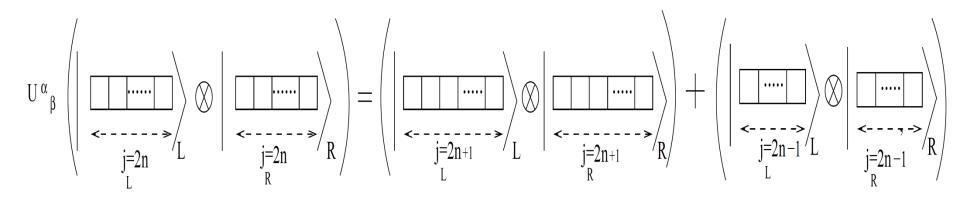
$$E_L^2(n,i) = E_R^2(n+i,i)$$

$$\Rightarrow a^{\dagger}(L) \bullet a(L) = a^{\dagger}(R) \bullet a(R)$$

#### **Features of Prepotential Formulation**

- ➤ Non-abelian fluxes can be absorbed locally at a site.
- > The abelian fluxes spread along the links.
- **>** Both the gauge symmetries SU(2)⊗U(1) together lead to non-local (involving at least a plaquette) Wilson loop states.

# Action of link operator



#### Consistent with U(1) Gauss law!

Local SU(2) fluxes are created at each end of the link connected by U(1) constraints

# SU(2) Gauge Theory Hilbert Space

 $\mathcal{H}_q$  is spanned by:

$$|j(n,i),m_L(n,i),m_R(n,i)\rangle = |j(n,i),m_L(n,i)\rangle_L \otimes |j(n,i),m_R(n,i)\rangle_R,$$

$$|j(n,i), m_L(n,i)\rangle_L = a_{\alpha_1}^{\dagger}(L)a_{\alpha_2}^{\dagger}(L)\cdots a_{\alpha_n}^{\dagger}(L)|0\rangle_L \equiv \hat{\mathcal{L}}_{\alpha_1\alpha_2\cdots\alpha_n}|0\rangle_L,$$
  

$$|j(n,i), m_R(n,i)\rangle_R = a_{\beta_1}^{\dagger}(R)a_{\beta_2}^{\dagger}(R)\cdots a_{\beta_n}^{\dagger}(R)|0\rangle_R \equiv \hat{\mathcal{R}}_{\beta_1\beta_2\cdots\beta_n}|0\rangle_R.$$

These operators are SU(2) irreducible as they are symmetric in all the SU(2) spin half indices.

#### Local SU(2) Gauge Invariant States

For d-dimensional lattice, 2d number of prepotential creation operators at each site all transforming in the same way under the SU(2) group present at the site



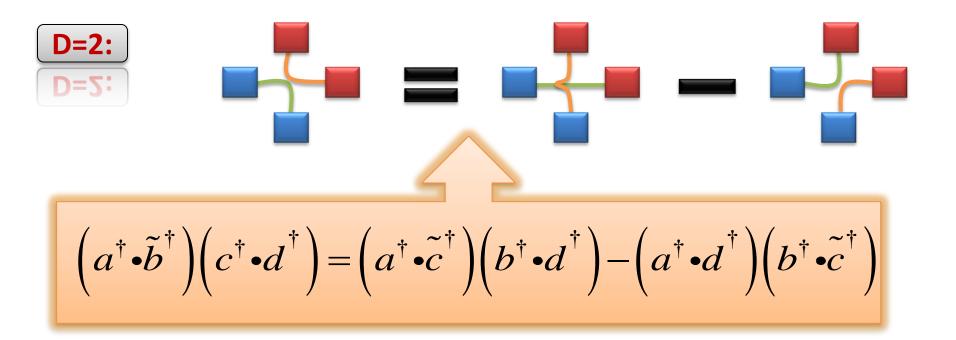
Hence all possible SU(2) invariant creation operators at site n are constructed by antisymmetrizing any two different prepotential doublets

$$L_{ij}(n) = \epsilon^{\alpha\beta} a_{\alpha}^{\dagger}(n,i) a_{\beta}^{\dagger}(n,j) = a^{\dagger}(n,i) \cdot \tilde{a}^{\dagger}(n,j), \qquad i,j = 1, 2, ..., 2d$$

Hence, the most general gauge invariant states at a lattice site n is given by,

$$|\vec{l}(n)\rangle = \prod_{i,j=1}^{2d} (L_{ij}(n))^{l_{ij}(n)} |0\rangle$$

#### SU(2) Local Gauge Invariant States & Mandelstam Constraints



Further, all possible mutually orthonormal loop states were explicitly constructed in terms of the prepotential operators. The dynamics of these orthonormal SU(2) loop states was shown to be governed by 3-nj Wigner coefficients.

# Prepotentials for SU(3)

$$a^{\dagger}(L)$$
 $n \leftarrow a^{\dagger}(R)$ 
 $a^{\dagger}(L)$ 
 $a^{\dagger}(L)$ 
 $a^{\dagger}(L)$ 
 $a^{\dagger}(R)$ 
 $a^{\dagger}(R)$ 

#### Abelian Gauss Law

Hilbert space of lattice gauge theory is built by applying the link

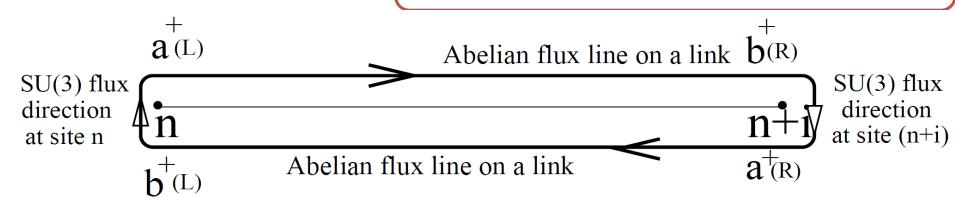
operators on the vacuum state:

 $U^{\alpha_1}{}_{\beta_1} U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_n}{}_{\beta_n} |0\rangle$ 

symmetrizing/antisymmetrizing the left  $\alpha \in 3$ indices automatically induces the same symmetries/antisymmetries on the right  $\beta \in 3*$  indices.

left and right representations are always conjugate to each other

$$\hat{N}(L) = \hat{M}(R), \quad \hat{M}(L) = \hat{N}(R)$$



# SU(3) Prepotential Hilbert space is characterized on every link by



$$|_{\alpha_{1}\alpha_{2}...\alpha_{p}}^{\beta_{1}\beta_{2}...\beta_{q}}\rangle_{L}\otimes|_{\gamma_{1}\gamma_{2}...\gamma_{q}}^{\delta_{1}\delta_{2}...\delta_{p}}\rangle_{R}\equiv\hat{L}_{\alpha_{1}\alpha_{2}...\alpha_{p}}^{\beta_{1}\beta_{2}...\beta_{q}}|0\rangle_{L}\otimes\hat{R}_{\gamma_{1}\gamma_{2}...\gamma_{q}}^{\delta_{1}\delta_{2}...\delta_{p}}|0\rangle_{R}$$



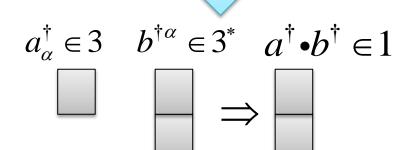
$$\hat{L}_{\alpha_1 \alpha_2 \cdots \alpha_p}^{\beta_1 \beta_2 \cdots \beta_q} |0\rangle_L \equiv a_{\alpha_1}^{\dagger}(L) \cdots a_{\alpha_p}^{\dagger}(L) b^{\dagger \beta_1}(L) \cdots b^{\dagger \beta_q}(L) |0\rangle_L$$

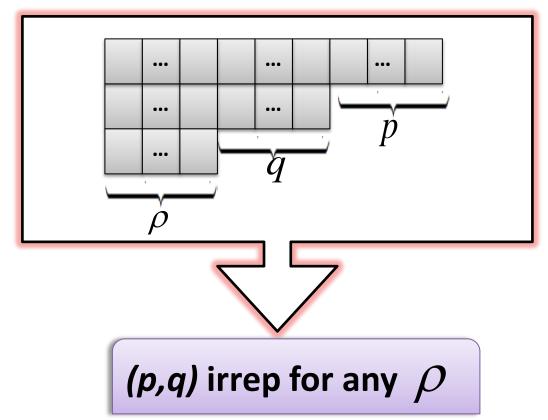
$$\hat{R}_{\gamma_1 \gamma_2 \cdots \gamma_q}^{\delta_1 \delta_2 \cdots \delta_p} |0\rangle_R \equiv a_{\gamma_1}^{\dagger}(R) \cdots a_{\gamma_q}^{\dagger}(R) b^{\dagger \delta_1}(R) \cdots b^{\dagger \delta_p}(R) |0\rangle_R$$

# SU(3) Reducible!!

Big difference with SU(2)

#### Origin of the problem





$$\mathcal{H}_{p} = \prod_{\otimes link} \left\{ \mathcal{H}_{p} \right\}_{link} = \prod_{\otimes link} \left\{ \sum_{\rho=0}^{\infty} \sum_{p,q=0}^{\infty} \left( \mathcal{H}_{p}^{L}(p,q,\rho) \otimes \mathcal{H}_{p}^{R}(q,p,\rho) \right) \right\}_{link}$$

$$\equiv \prod_{\otimes link} \left\{ \sum_{\rho=0}^{\infty} \mathcal{H}_{p}(\rho) \right\}_{link}.$$

As a consequence of  $\,E_L^2(n,i)=E_R^2(n+i,i)\,$  and U(1)XU(1) invariance

## SU(3) Gauge Theory Hilbert Space

the Hilbert space of SU(3) lattice gauge theory can be completely and uniquely labeled by

$$SU_L(3) \otimes Sp_L(2,R) \otimes SU_R(3) \otimes Sp_R(2,R)$$

quantum numbers on every link.

Generators of  $Sp_l(2,R)$ 

$$k_{-}(l) \equiv a(l) \cdot b(l)$$

$$k_{+}(l) \equiv a^{\dagger}(l) \cdot b^{\dagger}(l)$$

Satisfying,

$$k_0(l) \equiv \frac{1}{2} \left( a^{\dagger}(l) \cdot a(l) + b^{\dagger}(l) \cdot b(l) + 3 \right)$$

$$[k_0(l), k_{\pm}(l')] = \pm \delta_{l,l'} k_{\pm}(l), \quad [k_-(l), k_+(l')] = 2\delta_{l,l'} k_0(l)$$

$$|\mathcal{H}_p^R(q, p, \rho \pm 1)\rangle = k_{\pm}(R) |\mathcal{H}_p^R(q, p, \rho)\rangle |\mathcal{H}_p^L(p, q, \rho \pm 1)\rangle = k_{\pm}(L) |\mathcal{H}_p^L(p, q, \rho)\rangle$$

$$k_{-}(R) |\mathcal{H}_{p}^{R}(q, p, \rho = 0)\rangle = 0.$$
  $k_{-}(L) |\mathcal{H}_{p}^{L}(p, q, \rho = 0)\rangle = 0.$ 

$$\mathcal{H}_g \equiv \prod_{\text{olink}} \left\{ \mathcal{H}_p(\rho = 0) \right\}_{link} \equiv \mathcal{H}_p^0$$

# SU(3) Irreducible Prepotentials

$$[k_{-}(L), U^{\alpha}{}_{\beta}] \simeq 0, \quad [k_{-}(R), U^{\alpha}{}_{\beta}] \simeq 0,$$

We need to construct SU(3) irreducible prepotential operators from prepotential operators such that:

- 1. they have exactly the same SU(3)  $\otimes$  U(1)  $\otimes$  U(1) quantum numbers,
- 2. they commute with the Sp(2,R) destruction operator.

As a result, acting on the strong coupling vacuum they directly create states in the gauge theory Hilbert space.

# Irreducible Prepotential: Construction

$$A_{\alpha}^{\dagger}(L) = a_{\alpha}^{\dagger}(L) - F_L k_{+}(L)b_{\alpha}(L),$$
  

$$B^{\dagger \alpha}(L) = b^{\dagger \alpha}(L) - F_L k_{+}(L)a^{\alpha}(L),$$

$$A_{\alpha}^{\dagger}(R) = a_{\alpha}^{\dagger}(R) - F_R k_{+}(R)b_{\alpha}(R),$$
  

$$B^{\dagger \alpha}(R) = b^{\dagger \alpha}(R) - F_L k_{+}(R)a^{\alpha}(R).$$

$$F_L = \frac{1}{N(L) + M(L) + 1}, \quad F_R = \frac{1}{N(R) + M(R) + 1}.$$

These factors are chosen so that,

$$\left[k_{-}(l), A_{\alpha}^{\dagger}(l)\right] \simeq 0; \left[k_{-}(l), B^{\dagger \alpha}(l)\right] \simeq 0.$$

# We can now define states on a link in the gauge theory Hilbert space

$$|_{\alpha_{1}\alpha_{2}...\alpha_{p}}^{\beta_{1}\beta_{2}...\beta_{q}}\rangle_{L}^{0}\otimes|_{\gamma_{1}\gamma_{2}...\gamma_{q}}^{\delta_{1}\delta_{2}...\delta_{p}}\rangle_{R}^{0}\equiv\hat{\mathcal{L}}_{\alpha_{1}\alpha_{2}...\alpha_{p}}^{\beta_{1}\beta_{2}...\beta_{q}}|0\rangle_{L}\otimes\hat{\mathcal{R}}_{\gamma_{1}\gamma_{2}...\gamma_{q}}^{\delta_{1}\delta_{2}...\delta_{p}}|0\rangle_{R}.$$

With, 
$$\hat{\mathcal{L}}_{\alpha_{1}\alpha_{2}\cdots\alpha_{p}}^{\beta_{1}\beta_{2}\cdots\beta_{q}}|0\rangle_{L} \equiv A_{\alpha_{1}}^{\dagger}(L)\cdots A_{\alpha_{p}}^{\dagger}(L)B^{\dagger\beta_{1}}(L)\cdots B^{\dagger\beta_{q}}(L)|0\rangle_{L},$$
 
$$\hat{\mathcal{R}}_{\gamma_{1}\gamma_{2}\cdots\gamma_{q}}^{\delta_{1}\delta_{2}\cdots\delta_{p}}|0\rangle_{R} \equiv A_{\gamma_{1}}^{\dagger}(R)\cdots A_{\gamma_{q}}^{\dagger}(R)B^{\dagger\delta_{1}}(R)\cdots B^{\dagger\delta_{p}}(R)|0\rangle_{R}.$$

#### SU(3) Irreducible!

Having all the symmetries of SU(3) Young tableauex inbuilt

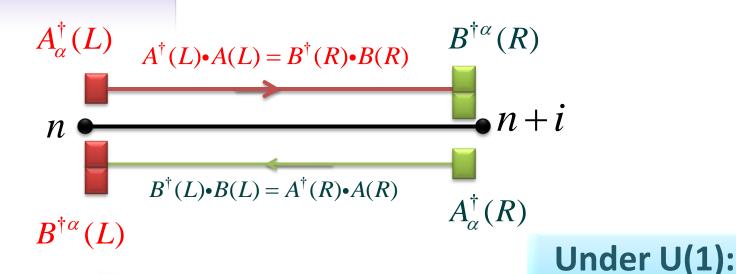
In terms of SU(3) irreducible prepotentials, the "spurious gauge invariant states" do not exist as:

$$A^{\dagger}(L) \cdot B^{\dagger}(L)|0\rangle_L \equiv 0,$$
  
 $A^{\dagger}(R) \cdot B^{\dagger}(R)|0\rangle_R \equiv 0.$ 

The role played by SU(2) prepotentials in SU(2) lattice gauge theory is exactly equivalent to the role played by SU(3) irreducible prepotentials in SU(3) lattice gauge theory.

## SU(3) Prepotential Formulation

#### $SU(3) \otimes U(1) \otimes U(1)$



#### Under SU(3):

$$A_{\alpha}^{\dagger}(L/R) \to A_{\beta}^{\dagger}(L/R) \left(\Lambda_{L/R}^{\dagger}\right)^{\beta}_{\alpha}$$
$$B^{\dagger \alpha}(L/R) \to \left(\Lambda_{L/R}^{\dagger}\right)^{\alpha}_{\beta} B^{\dagger \beta}(L/R)$$

$$A_{\alpha}^{\dagger}(L) \to e^{i\theta} A_{\alpha}^{\dagger}(L)$$

$$B^{\dagger \alpha}(R) \to e^{-i\theta} B^{\dagger \alpha}(R)$$

$$B^{\dagger \alpha}(L) \to e^{i\phi} B^{\dagger \alpha}(L)$$

$$A_{\alpha}^{\dagger}(R) \to e^{-i\phi} A_{\alpha}^{\dagger}(R)$$

# The Link Operator

#### Should have all the following group theoretical properties:

➤ Under SU(3) transformations

$$U(n,i)^{\alpha}{}_{\beta} \to (\Lambda_L)^{\alpha}{}_{\gamma}U(n,i)^{\gamma}{}_{\delta}(\Lambda_R^{\dagger})^{\delta}{}_{\beta}.$$

- $\triangleright$ It is invariant under U(1)  $\otimes$  U(1) abelian gauge transformations.
- $\succ$  It creates and destroys fluxes in  $\,{\cal H}^0_p$

$$U^{\alpha}{}_{\beta} = B^{\dagger \alpha}(L) \eta A^{\dagger}_{\beta}(R) + A^{\alpha}(L) \theta B_{\beta}(R) + \left(B(L) \wedge A^{\dagger}(L)\right)^{\alpha} \delta \left(A(R) \wedge B^{\dagger}(R)\right)_{\beta}.$$

$$U^{\alpha}{}_{\beta}|p,q\rangle_{L}\otimes|q,p\rangle_{R} = C_{1}{}^{\alpha}{}_{\beta}|p+1,q\rangle_{L}\otimes|q,p+1\rangle_{R} + C_{2}{}^{\alpha}{}_{\beta}|p,q-1\rangle_{L}\otimes|q-1,p\rangle_{R} + C_{3}{}^{\alpha}{}_{\beta}|p-1,q+1\rangle_{L}\otimes|q+1,p-1\rangle_{R}$$

#### Old vs. New Variables:

$$n = n + i$$

$$E_L^{a}(n,i) \qquad U(n,i) \qquad E_R^{a}(n+i,i)$$

$$A^{\dagger}(L) \frac{\lambda^{a}}{2} A(L) - B(L) \frac{\lambda^{a}}{2} B^{\dagger}(L)$$

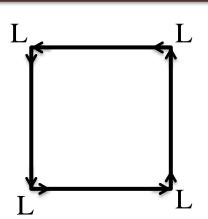
$$A^{\dagger}(R) \frac{\lambda^{a}}{2} A(R) - B(R) \frac{\lambda^{a}}{2} B^{\dagger}(R)$$

$$U = \underbrace{\left(\begin{array}{ccc} B^{\dagger 1}(L)\eta_L & A^1(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^1\delta_L \\ B^{\dagger 2}(L)\eta_L & A^2(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^2\delta_L \\ B^{\dagger 3}(L)\eta_L & A^3(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^3\delta_L \end{array}\right)}_{U_L} \underbrace{\left(\begin{array}{ccc} A^1(R)\bar{\eta}_R & B^{1\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^1\bar{\delta}_R \\ A^2(R)\bar{\eta}_R & B^{2\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^2\bar{\delta}_R \\ A^3(R)\bar{\eta}_R & B^{3\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^3\bar{\delta}_R \end{array}\right)^{\dagger}}_{U_R}$$

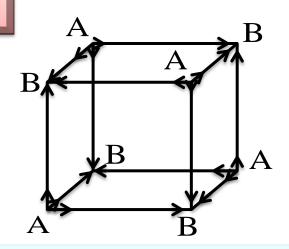
$$\eta_L = \frac{1}{\sqrt{B(L)\cdot B^{\dagger}(L)}}, \quad \theta_L = \frac{1}{\sqrt{A^{\dagger}(L)\cdot A(L)}}, \quad \delta_L = \frac{1}{\sqrt{(A(L)\wedge B^{\dagger}(L))\cdot (B(L)\wedge A^{\dagger}(L))}};$$

$$\eta_R = \frac{1}{\sqrt{A^{\dagger}(R)\cdot A(R)}}, \quad \theta_R = \frac{1}{\sqrt{B(R)\cdot B^{\dagger}(R)}}, \quad \delta_R = \frac{1}{\sqrt{(A(R)\wedge B^{\dagger}(R))\cdot (B(R)\wedge A^{\dagger}(R))}}.$$

## **Local Gauge Invariant States**



$$\mathbf{L}_{[ij]} = A^{\dagger} [i] \cdot B^{\dagger} [j],$$



$$\mathbf{B}_{\left[j_{1}j_{2}j_{3}\right]}=B^{\dagger}\left[j_{1}\right]\wedge B^{\dagger}\left[j_{2}\right]\wedge B^{\dagger}\left[j_{3}\right]$$

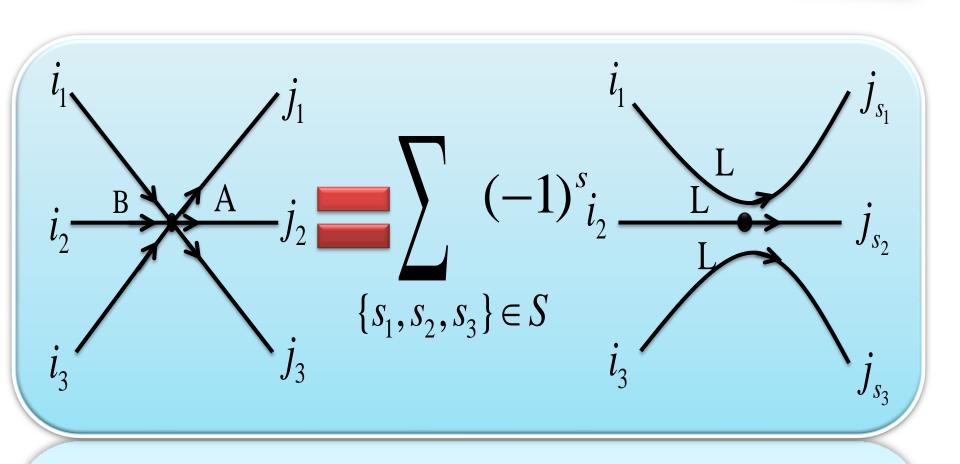
$$\mathbf{A}_{\left[i_{1}i_{2}i_{3}\right]}=A^{\dagger}\left[i_{1}\right]\wedge A^{\dagger}\left[i_{2}\right]\wedge A^{\dagger}\left[i_{3}\right]$$

$$|\vec{l}_{[ij]}, \vec{p}_{[i_1i_2i_3]}, \vec{q}_{[j_1j_2j_3]}\rangle =$$

$$\prod_{\substack{i,j=1\\i\neq j}}^{2d} \left(L_{[ij]}\right)^{l_{[ij]}} \prod_{\substack{j=1\\[i_1i_2i_3]=1}}^{2d} \left(A_{[i_1i_2i_3]}\right)^{p_{[i_1i_2i_3]}} \prod_{\substack{j=1\\[i_1j_2j_3]=1}}^{2d} \left(B_{[j_1j_2j_3]}\right)^{q_{[j_ji_2j_3]}} |0\rangle.$$

#### SU(3) Mandelstam Constraints:

## The Simplest One



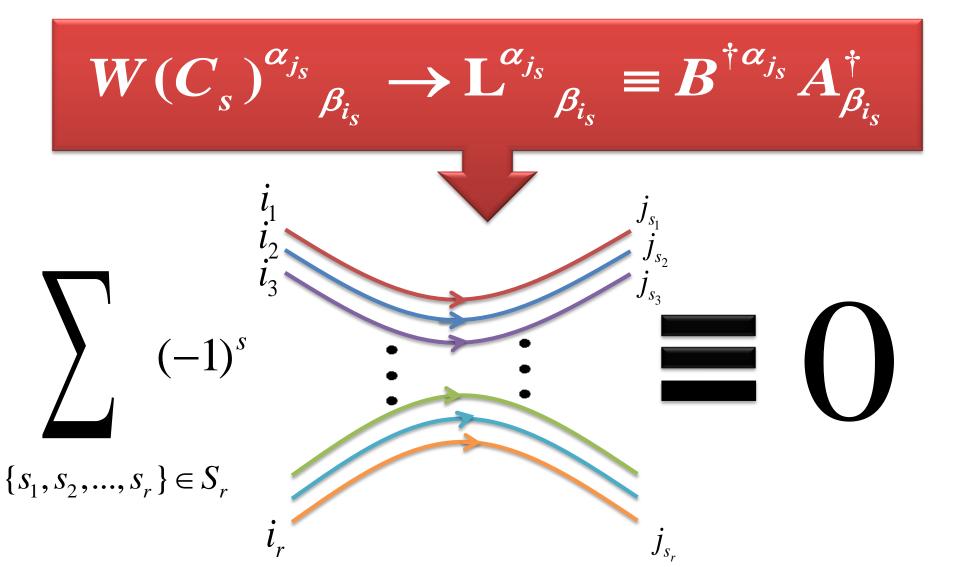
#### Mandelstam Constraints for Wilson loop

FOR r number of arbitrary loops  $C_r$  (r>3) with any possible shapes and sizes starting from a site n to  $i_1, i_2, ..., i_r$  directions and coming back at n from  $j_1, j_2, ..., j_r$  directions,

Tr 
$$W(C_1)$$
Tr  $W(C_2)$ ...Tr  $W(C_r)$   
-Tr  $W(C_1C_2)$ Tr  $W(C_3)$ ...Tr  $W(C_r)$   
+....+ $(-1)^{r+1}$ Tr  $W(C_1C_2...C_r) \equiv 0$ 

#### **Non Local Constraints!!**

# Advantage of Prepotentials: All Constraints are Local



## Summary:

- Prepotentials: Transforming Like Fundamental Matter Fields.
- ➤ SU(3) irreducible prepotentials: directly creates QCD fluxes around lattice sites.
- > SU(3) invariant vertices at every lattice site.

ALL SU(3)
MANDELSTAM
CONSTRAINTS IN
LOCAL FORMS.

Prepotential
Formulation for
Higher SU(N)
Lattice Gauge
Theory



Solve all the Mandelstam Constraints for SU(3)

# Thank You