Tata Lectures on Overlap Fermions

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March 14-17, 2011

TIFR, Mumbai, India

Overview

- Introduction
- Abelian gauge fields in two dimensions
- Wilson fermions
- Pauli-Villars regularization of chiral fermions
- Overlap formalism
- Chiral anomalies

Chiral Gauge Theories

- Strong interactions
  - Vector like gauge theory
  - Spontaneous chiral symmetry breaking
  - Axial anomaly and instantons
- Electroweak interactions
  - Chiral gauge theory
  - Anomaly cancellation
  - ‘t Hooft vertex

Abelian gauge theories in two dimensions

- Schwinger model (like strong)
- Chiral Schwinger model (like Electroweak)
- Clean split of the lattice gauge fields into
  - Torons
  - Gauge orbit representative
  - Gauge transformations
  - Topological piece
- Fermionic index

$$Z = \int [dA] e^{-S_g(A)} \int [d\psi][d\overline{\psi}] e^{S_f(\overline{\psi}, \psi, A)}$$

$$= \int [dA] e^{-S_g(A) - W_f(A)}$$

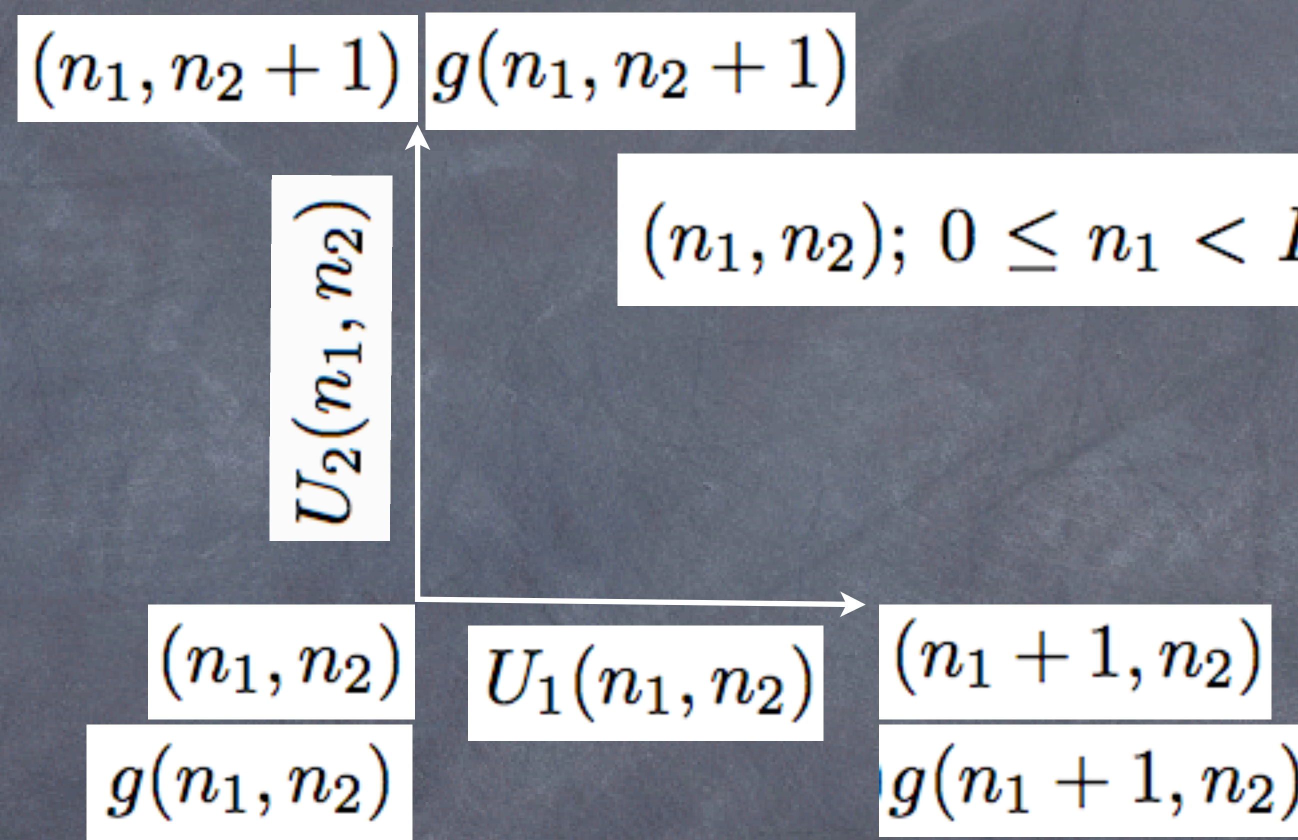
$$e^{-W_f(A)} = \int [d\psi][d\overline{\psi}] e^{S_f(\overline{\psi}, \psi, A)}$$

Abelian gauge fields on a 2d lattice

- Finite lattice
- Periodic boundary conditions

$$(n_1, n_2);\ 0 \le n_1 < L_1 \text{ and } 0 \le n_2 < L_2$$

$$L_1 \times L_2$$



$$\begin{aligned}
U_1(n_1, n_2) &\to U_1'(n_1, n_2) = g^\dagger(n_1, n_2) U_1(n_1, n_2) g(n_1+1, n_2) \\
U_2(n_1, n_2) &\to U_2'(n_1, n_2) = g^\dagger(n_1, n_2) U_2(n_1, n_2) g(n_1, n_2+1)
\end{aligned}$$

$(0,0)$

(n_1, n_2+1) $\quad U_1^\dagger(n_1, n_2+1)$

$U_2^\dagger(n_1, n_2)$ **Plaquette** $U_2(n_1+1, n_2)$

(n_1, n_2) $\quad U_1(n_1, n_2)$ $\quad (n_1+1, n_2)$

Toron

$$e^{i\pi h_2} = \prod_{n_2=0}^{L_2-1} U_2(0, n_2)$$

$$e^{iE(n_1,n_2)} = U_1(n_1,n_2)U_2(n_1+1,n_2)U_1^\dagger(n_1,n_2+1)U_2^\dagger(n_1,n_2)$$

$$-\pi < E(n_1,n_2) \leq \pi$$

Toron

$(0,0)$ $\quad (0,0)$

$$e^{i\pi h_1} = \prod_{n_1=0}^{L_1-1} U_1(n_1, 0)$$

Continuum limit

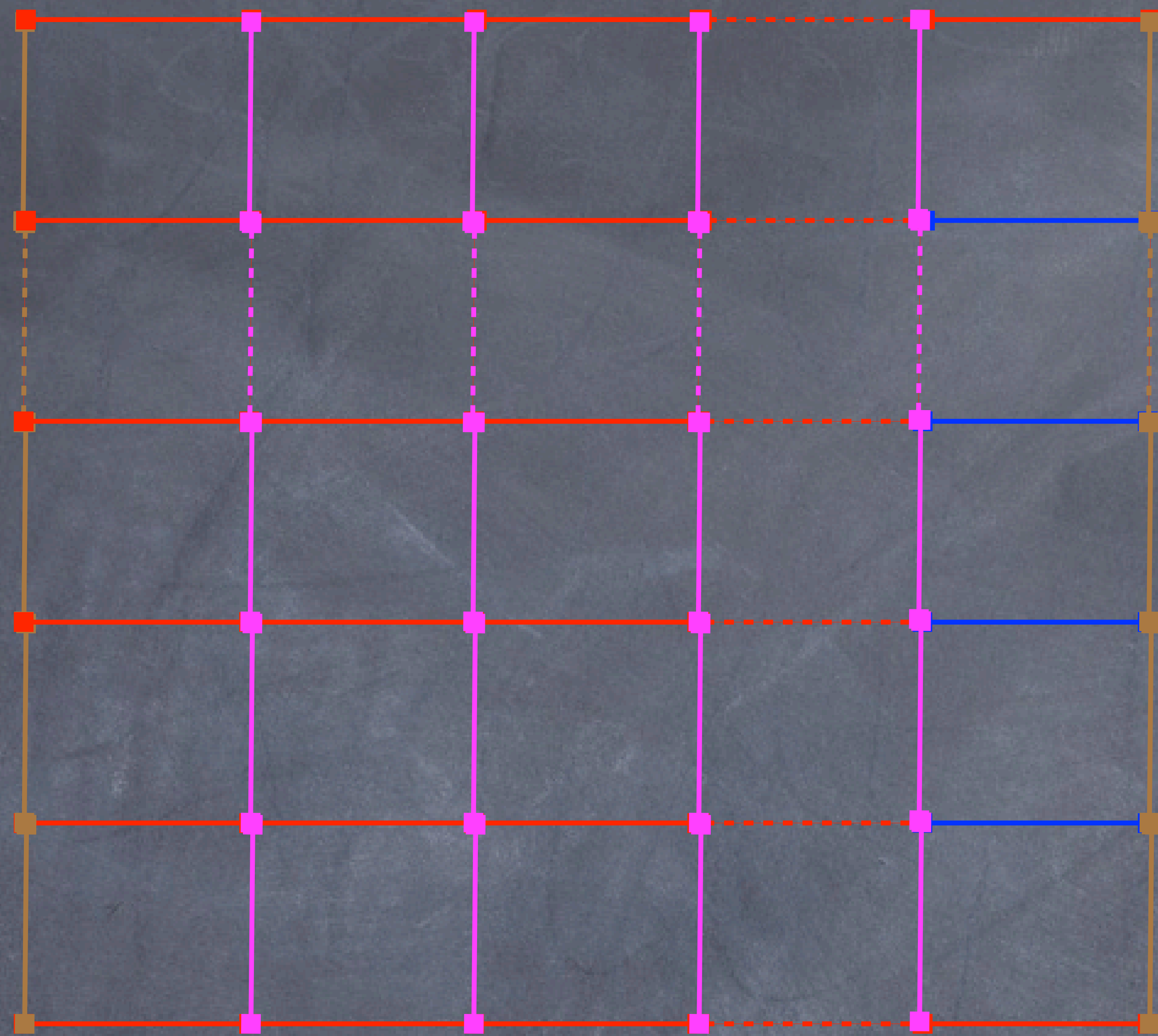
$$S_g = -\beta \sum_{n_1,n_2} \cos E(n_1,n_2)$$

$$\beta, L_1, L_2 \to \infty$$

$$\frac{L_1}{\sqrt{\beta}} = l_1 \text{ and } \frac{L_2}{\sqrt{\beta}} = l_2 \text{ fixed.}$$

$$U_\mu(n_1,n_2) \to e^{\frac{i}{\sqrt{\beta}} A_\mu(x_1,x_2)}; \qquad x_\mu = \frac{n_\mu}{\sqrt{\beta}}$$

$$\beta E(n_1,n_2) \to e(x_1,x_2)$$



$e^{i\frac{\pi h_1}{L_1}}$

$e^{i\frac{\pi h_2}{L_2}}$

$e^{i\sum_{k_1=0}^{n_1-1} E(k_1,n_2)} e^{i\frac{\pi h_2}{L_2}}$

(n_1, n_2)

$e^{-i\sum_{k_2=0}^{n_2-1}\sum_{k_1=0}^{L_1-1} E(k_1,k_2)} e^{i\frac{\pi h_1}{L_1}}$

$(L_1 - 1, n_2)$

Top Right Plaquette

$$\sum_{k_1=0}^{L_1-1}\sum_{k_2=0}^{L_2-1} E(k_1,k_2) = 2\pi Q$$

Q is an integer

Q: Topological charge

Given

$$E(n_1, n_2) \qquad h_1 \text{ and } h_2$$

solve

$$\Box\phi(n_1, n_2) = E(n_1, n_2) - \frac{2\pi Q}{L_1 L_2}$$

where

$$\Box\phi(n_1, n_2) = -4\phi(n_1, n_2) + \phi(n_1 - 1, n_2) + \phi(n_1 + 1, n2) + \phi(n_1, n_2 - 1) + \phi(n_1, n_2 + 1)$$

then

$$U_1(n_1, n_2) = e^{i\frac{\pi h_1}{L_1}} U_1^Q(n_1, n_2) e^{-i\chi(n_1,n_2)} e^{i[\phi(n_1,n_2-1)-\phi(n_1,n_2)]} e^{i\chi(n_1+1,n_2)}$$
$$U_2(n_1, n_2) = e^{i\frac{\pi h_2}{L_2}} U_2^Q(n_1, n_2) e^{-i\chi(n_1,n_2)} e^{i[\phi(n_1,n_2)-\phi(n_1-1,n_2)]} e^{i\chi(n_1,n_2+1)}$$

$$U_1^Q(n_1, n_2) = e^{-i\frac{2\pi Q}{L_1 L_2} n_2}; \quad 0 \le n_1 < L_1; \quad 0 \le n_2 < L_2;$$
$$U_2^Q(n_1, n_2) = \begin{cases} 1 & \text{if } 0 \le n_1 < L_1 \text{ and } 0 \le n_2 < L_2 - 1 \\ e^{i\frac{2\pi Q}{L_1} n_1} & \text{if } 0 \le n_1 < L_1 \text{ and } n_2 = L_2 \end{cases}$$

Continuum limit

$$A_1(x_1,x_2) = \frac{\pi h_1}{l_1} + \partial_1\chi(x_1,x_2) - \partial_2\phi(x_1,x_2) - \frac{2\pi Q}{l_1 l_2}x_2$$
$$A_2(x_1,x_2) = \frac{\pi h_2}{l_2} + \partial_2\chi(x_1,x_2) + \partial_1\phi(x_1,x_2).$$

$$A_1(x_1,0) \neq A_1(x_1,l_2)$$

$$A_1(x_1,l_2) = A_1(x_1,0) - \partial_1\frac{2\pi Q x_1}{l_1}$$

non-trivial winding

$$\Box\phi(x_1,x_2) = \left(\partial_1^2 + \partial_2^2\right)\phi(x_1,x_2) = e(x_1,x_2) - \frac{2\pi Q}{l_1 l_2}$$

Massless Dirac operator - Topological Zero modes

$$\not{D} = \sum_{\mu=1}^{2} \sigma_\mu (\partial_\mu + iA_\mu)$$ $$\not{D}\sigma_3 = -\sigma_3 \not{D}$$ $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\not{D} = f^* e^{-\sigma_3\phi} \not{D}_Q f e^{-\sigma_3\phi}$$ $$f = e^{i\frac{\pi h_1 x_1}{l_1} + i\frac{\pi h_2 x_2}{l_2} + i\chi}$$ $$\not{D}_Q = \begin{pmatrix} 0 & C_Q \\ -C_Q^\dagger & 0 \end{pmatrix}; \quad C_Q = \partial_1 - i\partial_2 - ibx_2; \quad b = \frac{2\pi Q}{l_1 l_2}$$

$$-\not{D}_Q^2 = \begin{pmatrix} C_Q C_Q^\dagger & 0 \\ 0 & C_Q^\dagger C_Q \end{pmatrix} = \begin{pmatrix} K-b & 0 \\ 0 & K+b \end{pmatrix}$$ $$K = -\partial_1^2 - \partial_2^2 + 2ibx_2\partial_1 + b^2 x_2^2$$

$$K \pm b\psi = 0$$ $$\psi(x_1 + l_1, x_2) = \psi(x_1, x_2); \quad \psi(x_1, x_2 + l_2) = e^{i\frac{2\pi Q}{l_1}x_1}\psi(x_1, x_2)$$

$$\psi(x_1, x_2) = \sum_{p=-\infty}^{\infty} a_p e^{i\frac{2\pi p}{l_1}x_1} h\left(x_2 - \frac{pl_2}{Q}\right)$$ $$\left(-\partial_y^2 + b^2y^2 \pm b\right) h(y) = 0$$ $$h(y) = e^{-\frac{|b|}{2}y^2}$$

$$\psi(x_1, x_2) = \sum_{p=-\infty}^{\infty} a_p e^{i\frac{2\pi p}{l_1}x_1} e^{-\frac{\pi|Q|}{l_1 l_2}\left(x_2 - \frac{pl_2}{Q}\right)^2}$$ $$\psi(x_1, x_2 + l_2) = \sum_{p=-\infty}^{\infty} a_{p+Q} e^{i\frac{2\pi(p+Q)}{l_1}x_1} e^{-\frac{\pi|Q|}{l_1 l_2}\left(x_2 - \frac{pl_2}{Q}\right)^2}$$

$$a_{p+Q} = a_p$$

$$\psi_k(z_1, z_2) = \frac{1}{\sqrt{l_1 l_2}} e^{2\pi i k z_1 - \pi|Q|\tau\left(z_2 - \frac{k}{Q}\right)^2} \vartheta\left(Q\left(z_1 + i\tau z_2\right) - ik\tau; i|Q|\tau\right)$$

$k = 0, \cdots, |Q-1|$ and $z_\mu = \frac{x_\mu}{l_\mu}$, $\tau = \frac{l_2}{l_1}$

Some remarks

- Q=0
  - There are no zero modes.
  - The determinant of $\not{D}_Q$ is that of free fermions.
  - There is a Jacobian in the determinant of $\not{D}$.
  - The Jacobian is formally unity but it needs to be computed using a proper regularization scheme.
- $Q \neq 0$
  - There are chiral zero modes.
  - The determinant is formally zero.
  - Certain fermion number violating observables are non zero.
  - Can compute all eigenvalues of $\not{D}_Q$.

$$E(n_1,n_2)-\frac{2\pi Q}{L_1L_2}=\frac{1}{L_1L_2}\sum_{p_1=-\left\lfloor\frac{L_1-1}{2}\right\rfloor}^{\left\lceil\frac{L_1-1}{2}\right\rceil}\sum_{p_2=-\left\lfloor\frac{L_2-1}{2}\right\rfloor}^{\left\lceil\frac{L_2-1}{2}\right\rceil}\tilde{E}(p_1,p_2)e^{i\frac{2\pi p_1 n_1}{L_1}+i\frac{2\pi p_2 n_2}{L_2}}$$

Discrete Fourier Transform

$$\phi(n_1,n_2)=\frac{1}{L_1L_2}\sum_{p_1=-\left\lfloor\frac{L_1-1}{2}\right\rfloor}^{\left\lceil\frac{L_1-1}{2}\right\rceil}\sum_{p_2=-\left\lfloor\frac{L_2-1}{2}\right\rfloor}^{\left\lceil\frac{L_2-1}{2}\right\rceil}\tilde{\phi}(p_1,p_2)e^{i\frac{2\pi p_1 n_1}{L_1}+i\frac{2\pi p_2 n_2}{L_2}}$$

Brillouin Zone

$$p_\mu\in\left[-\left\lfloor\frac{L_\mu-1}{2}\right\rfloor,\left\lceil\frac{L_\mu-1}{2}\right\rceil\right]$$

$$\tilde{\phi}(p_1,p_2)=\begin{cases}0 & \text{if } p_1=p_2=0\\ -\frac{\tilde{E}(p_1,p_2)}{4\left[\sin^2\left(\frac{\pi p_1}{L_1}\right)+\sin^2\left(\frac{\pi p_2}{L_2}\right)\right]} & \text{otherwise}\end{cases}$$

only one pole

$$\not{D}\psi(n_1,n_2)=\sigma_1[\psi(n_1+1,n_2)-\psi(n_1-1,n_2)]+\sigma_2[\psi(n_1,n_2+1)-\psi(n_1,n_2-1)]$$

4 zeros

$$\not{D}\tilde{\psi}(p_1,p_2)=2i\left[\sigma_1\sin\left(\frac{2\pi p_1}{L_1}\right)+\sigma_2\sin\left(\frac{2\pi p_2}{L_2}\right)\right]\tilde{\psi}(p_1,p_2)$$

3 doublers

Wilson fermions $\not{D}_w=\not{D}-\Box$ $-\Box$ mass for doublers

But $\sigma_3\not{D}\sigma_3=-\not{D}$ not obeyed by $\not{D}_w$
(chiral symmetry)

Not a solution for
$\not{C}=\not{D}P_+$ $P_\pm=\frac{1\pm\sigma_3}{2}$

$$S_{+f}(\overline{\psi}_+, \psi_+, A) = \int d^2x \overline{\psi}_+ \partial \psi_+ + iq_+ \int d^2x \overline{\psi}_+ A \psi_+$$

$$\partial = \partial_1 + i\partial_2$$

$$A = A_1 + iA_2$$

$$S_{-f}(\overline{\psi}_-, \psi_-, A) = \int d^2x \overline{\psi}_- \overline{\partial} \psi_- + iq_- \int d^2x \overline{\psi}_- \overline{A} \psi_-$$

$$e^{-W_+(A)} = \int [d\overline{\psi}][d\psi] e^{S_{+f}(\overline{\psi}_+, \psi_+, A)}$$

$h_\mu = 0$ $\qquad$ $Q = 0$

$$A = \partial \xi; \quad \xi = \chi + i\phi.$$

$$\psi_+ \to \psi'_+ = e^{i\xi}\psi_+; \quad \overline{\psi}_+ \to \overline{\psi}'_+ = e^{-i\xi}\overline{\psi}_+; \quad \psi_- \to \psi'_- = e^{i\bar{\xi}}\psi_-; \quad \overline{\psi}_- \to \overline{\psi}'_- = e^{-i\bar{\xi}}\overline{\psi}_-$$

Fermions decouple and dependence on gauge field can only come from the Jacobian

Jacobian is not unity since there are divergences

$$\psi(x) = \frac{1}{\sqrt{l_1 l_2}} \sum_{p_\mu=-\infty}^{\infty} \tilde{\psi}(p) e^{i\frac{2\pi p_1 x_1}{l_1} + i\frac{2\pi p_2 x_2}{l_2}}$$
$$\overline{\psi}(x) = \frac{1}{\sqrt{l_1 l_2}} \sum_{p_\mu=-\infty}^{\infty} \overline{\tilde{\psi}}(p) e^{-i\frac{2\pi p_1 x_1}{l_1} - i\frac{2\pi p_2 x_2}{l_2}}$$

$$A(x) = \sum_{p_\mu=-\infty}^{\infty} \tilde{A}(p) e^{i\frac{2\pi p_1 x_1}{l_1} + i\frac{2\pi p_2 x_2}{l_2}}$$

$$\langle \tilde{\psi}_+(q) \overline{\tilde{\psi}}_+(p) \rangle = -\frac{\delta_{pq}}{\mathcal{P}}, \quad \mathcal{P} = 2\pi i \left(\frac{p_1}{l_1} + i \frac{p_2}{l_2} \right)$$

$$S_I = i q_+ \sum_{p_\mu, q_\mu=-\infty}^{\infty} \overline{\tilde{\psi}}_+(p+q) \sigma_\mu \tilde{A}_\mu(p) \tilde{\psi}_+(q)$$

$$W_+(A_\mu) = \sum_{k=1}^{\infty} (-i q_+)^k \sum_{p^j_\mu=-\infty}^{\infty} \left[\prod_{j=1}^{k} \tilde{A}(p^j) \right] \sum_{r_\mu=-\infty}^{\infty} \frac{1}{\mathcal{R}} \frac{1}{\mathcal{R}+\mathcal{P}^1} \frac{1}{\mathcal{R}+\mathcal{P}^1+\mathcal{P}^2} \cdots \frac{1}{\mathcal{R}+\mathcal{P}^1+\mathcal{P}^2+\cdots+\mathcal{P}^{k-1}} \delta\left(\sum_{j=1}^{k} p^j \right). \tag{4.9}$$

Sum over simple poles
Sum of residues is zero
Shift r sum
Result is zero unless sum is divergent

Only k=2 can contribute

Pauli-Villars regularization

- One set for PV fields for each chiral fermion
- Show that result is finite if $$\sum_{i=1}^{n_+} (q_+^i)^2 = \sum_{j=1}^{n_-} \left(q_-^j\right)^2$$
- If $n_+ = n_-$ and $q_+^i = q_-^i$ for all i, QCD like and one can use finite number of PV fields and regulate the theory

PV fields are massive and each PV field has twice the number of degrees of freedom compared to a single chiral field.

Need infinite number of PV fields to regulate one chiral fermion

$$e^{-W_+(A_\mu)} = \int [d\overline{\psi}][d\psi] e^{\int d^2x \overline{\psi}[\sigma_\mu(\partial_\mu + iq_+ A_\mu) + P_+ M + P_- M^\dagger]\psi}$$

M Infinite mass matrix

M has a zero mode

$M^\dagger$ has no zero modes

Under parity $x_1 \to -x_1$ and $x_2 \to x_2$

$$A_1(x_1, x_2) \to -A_1(-x_1, x_2); \quad A_2(x_1, x_2) \to A_2(-x_1, x_2),$$

$$\psi(x_1, x_2) \to \sigma_2 \psi(-x_1, x_2); \quad \overline{\psi}(x_1, x_2) \to \overline{\psi}(-x_1, x_2)\sigma_2;$$

$$M \to M^\dagger$$

$+ \Rightarrow -$

$$S_q = \int d^2x \, (\,\overline{\psi}_+(x) \quad \overline{\psi}_-(x)\,) \begin{pmatrix} \partial & M^\dagger \\ M & \bar{\partial} \end{pmatrix} \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$$

$$S_q = \sum_{p_\mu=-\infty}^{\infty} (\,\bar{\tilde{\psi}}_+(p) \quad \bar{\tilde{\psi}}_-(p)\,) \begin{pmatrix} \mathcal{P}_1 + i\mathcal{P}_2 & M^\dagger \\ M & \mathcal{P}_1 - i\mathcal{P}_2 \end{pmatrix} (\,\tilde{\psi}_+(p) \quad \tilde{\psi}_+(p)\,)$$

$$\mathcal{P}_\mu = 2\pi i \frac{p_\mu}{l_\mu} \qquad\qquad |\mathcal{P}|^2 = -\mathcal{P}_1^2 - \mathcal{P}_2^2$$

$$\langle \tilde{\psi}_+(p)\bar{\tilde{\psi}}_+(p)\rangle = \frac{\mathcal{P}_1 - i\mathcal{P}_2}{|\mathcal{P}|^2 + M^\dagger M}, \;\; \langle \tilde{\psi}_-(p)\bar{\tilde{\psi}}_-(p)\rangle = \frac{\mathcal{P}_1 + i\mathcal{P}_2}{|\mathcal{P}|^2 + MM^\dagger},$$
$$\langle \tilde{\psi}_-(p)\bar{\tilde{\psi}}_+(p)\rangle = -M\frac{1}{|\mathcal{P}|^2 + M^\dagger M}, \;\; \langle \tilde{\psi}_+(p)\bar{\tilde{\psi}}_-(p)\rangle = -M^\dagger\frac{1}{|\mathcal{P}|^2 + MM^\dagger},$$

$$S_I = iq_+ \sum_{p_\mu, q_\mu=-\infty}^{\infty} \left[\bar{\tilde{\psi}}_+(p+q)\tilde{A}(p)\tilde{\psi}_+(q) + \bar{\tilde{\psi}}_-(p+q)\bar{\tilde{A}}(p)\tilde{\psi}_-(q)\right]$$

A specific example -- Harmonic Oscillator

$$M_{jk} = \Lambda k \delta_{j,k-1}; \quad M^\dagger_{jk} = \Lambda j \delta_{j,k+1};$$

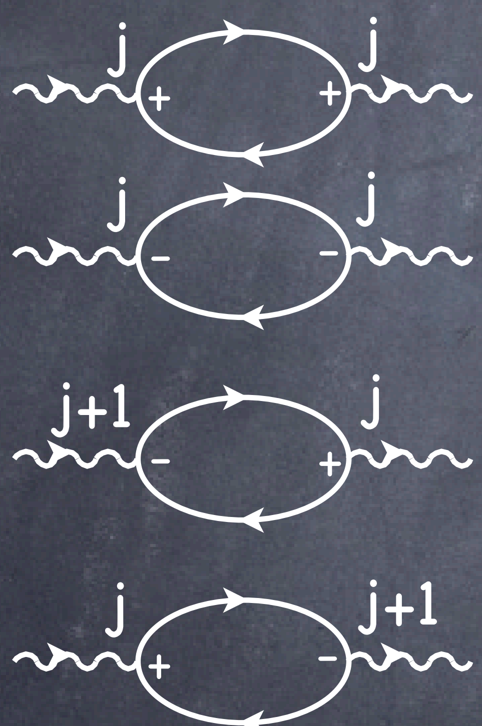
$$v_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}; \quad Mv = 0,$$

$$(MM^\dagger)_{jk} = \Lambda^2 (j+1)^2 \delta_{jk}; \quad (M^\dagger M)_{jk} = \Lambda^2 j^2 \delta_{jk}.$$

$$G^{++}_{jj}(p) = \frac{\mathcal{P}_1 - i\mathcal{P}_2}{\mathcal{P}^2 + \Lambda^2 j^2} \quad G^{--}_{jj}(p) = \frac{\mathcal{P}_1 + i\mathcal{P}_2}{\mathcal{P}^2 + \Lambda^2 (j+1)^2};$$
$$G^{-+}_{j(j+1)}(p) = -\frac{\Lambda(j+1)}{\mathcal{P}^2 + \Lambda^2 (j+1)^2}; \quad G^{+-}_{(j+1)j}(p) = -\frac{\Lambda(j+1)}{\mathcal{P}^2 + \Lambda^2 (j+1)^2};$$

spin-statistics

$$\overline{\psi}_{+j}; \quad \psi_{+j} : \begin{cases} \text{fermion for even } j \\ \text{boson for odd } j \end{cases};$$
$$\overline{\psi}_{-j}; \quad \psi_{-j} : \begin{cases} \text{fermion for odd } j \\ \text{boson for even } j \end{cases}.$$



$$
\begin{aligned}
W_+(A_\mu) = q_+^2 \sum_{p_\mu=-\infty}^{\infty} \\
&\tilde{A}(p)\tilde{A}(-p) \sum_{q_\mu=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j G^{++}_{jj}(q-\frac{p}{2})G^{++}_{jj}(q+\frac{p}{2}) \\
&+\bar{\tilde{A}}(p)\bar{\tilde{A}}(-p) \sum_{q_\mu=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+1} G^{--}_{jj}(q-\frac{p}{2})G^{--}_{jj}(q+\frac{p}{2}) \\
&+\tilde{A}(p)\bar{\tilde{A}}(-p) \sum_{q_\mu=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j G^{+-}_{j(j+1)}(q-\frac{p}{2})G^{-+}_{(j+1)j}(q+\frac{p}{2}) \\
&+\bar{\tilde{A}}(p)\tilde{A}(-p) \sum_{q_\mu=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j G^{-+}_{(j+1)j}(q-\frac{p}{2})G^{+-}_{j(j+1)}(q+\frac{p}{2})
\end{aligned}
$$

$$q_+^2\Big\{\Big[\tilde{A}_1(p)\tilde{A}_1(-p)-\tilde{A}_2(p)\tilde{A}_2(-p)\Big]\left[(\mathcal{Q}_1-\tfrac{\mathcal{P}_1}{2})(\mathcal{Q}_1+\tfrac{\mathcal{P}_1}{2})-(\mathcal{Q}_2-\tfrac{\mathcal{P}_2}{2})(\mathcal{Q}_2+\tfrac{\mathcal{P}_2}{2})\right]$$

$$+\Big[\tilde{A}_1(p)\tilde{A}_2(-p)+\tilde{A}_2(p)\tilde{A}_1(-p)\Big]\left[(\mathcal{Q}_1-\tfrac{\mathcal{P}_1}{2})(\mathcal{Q}_2+\tfrac{\mathcal{P}_2}{2})+(\mathcal{Q}_2-\tfrac{\mathcal{P}_2}{2})(\mathcal{Q}_1+\tfrac{\mathcal{P}_1}{2})\right]\Big\}$$

$$\textstyle\sum_{j=-\infty}^{\infty}\frac{(-1)^j}{\left[\left(\mathcal{Q}-\frac{\mathcal{P}}{2}\right)^2+\Lambda^2j^2\right]\left[\left(\mathcal{Q}+\frac{\mathcal{P}}{2}\right)^2+\Lambda^2j^2\right]}$$

$$+\,iq_+^2\Big\{\Big[\tilde{A}_2(p)\tilde{A}_2(-p)-\tilde{A}_1(p)\tilde{A}_1(-p)\Big]\left[(\mathcal{Q}_1-\tfrac{\mathcal{P}_1}{2})(\mathcal{Q}_2+\tfrac{\mathcal{P}_2}{2})+(\mathcal{Q}_2-\tfrac{\mathcal{P}_2}{2})(\mathcal{Q}_1+\tfrac{\mathcal{P}_1}{2})\right]$$

$$+\Big[\tilde{A}_1(p)\tilde{A}_2(-p)+\tilde{A}_2(p)\tilde{A}_1(-p)\Big]\left[(\mathcal{Q}_1-\tfrac{\mathcal{P}_1}{2})(\mathcal{Q}_1+\tfrac{\mathcal{P}_1}{2})-(\mathcal{Q}_2-\tfrac{\mathcal{P}_2}{2})(\mathcal{Q}_2+\tfrac{\mathcal{P}_2}{2})\right]\Big\}$$

$$\frac{1}{\left[\left(\mathcal{Q}-\frac{\mathcal{P}}{2}\right)^2\right]\left[\left(\mathcal{Q}+\frac{\mathcal{P}}{2}\right)^2\right]} \qquad (4.24)$$

Real part $$\sum_{j=-\infty}^{\infty}\frac{(-1)^j}{\mathcal{R}^2+\Lambda^2j^2}=\frac{\pi}{\Lambda|\mathcal{R}|\sinh\left(\frac{\pi|\mathcal{R}|}{\Lambda}\right)},$$ well regulated

Imaginary part $$\sum_{i=1}^{n_+}\left(q_+^i\right)^2=\sum_{j=1}^{n_-}\left(q_-^j\right)^2 \Rightarrow$$ zero

Move from the eigenbasis of Harmonic oscillator to coordinates

$$M = \frac{\Lambda}{2}(\partial_s + s)\sqrt{-\partial_s^2 + s^2 - 1}.$$

- The fermionic action is quadratic
- Go to the second quantized representation
- We have a s-dependent Hamiltonian

Use $$M = \partial_s + m(s); \quad m(s) = \begin{cases} \Lambda & \text{for } s > 0 \\ -m & \text{for } s \le 0 \end{cases}; \quad m, \Lambda > 0.$$

M has a zero mode $$\phi(s) = \begin{cases} e^{-\Lambda s} & \text{for } s \ge 0 \\ e^{ms} & \text{for } s \le 0 \end{cases}$$ $M^\dagger$ does not have a zero mode

Two many body Hamiltonians

$$\mathcal{H}^{\pm} = a^\dagger H^{\pm} a; \quad H^{\pm} = \sigma_3 \left[\sum_{\mu=1}^{2} \sigma_\mu \left(\partial_\mu + iA_\mu\right) + m^{\pm} \right]; \quad m^+ = \Lambda; \quad m^- = -m.$$

Let $\Lambda_\pm$ and $|\pm\rangle$

be the highest eigenvalues and the corresponding eigenvectors of $\mathcal{H}_\pm$

a-type fermions

$$m(s) = \begin{cases} \Lambda & \text{for } s > 0 \\ -m & \text{for } s \leq 0 \end{cases}$$

b-type fermions

$$m(s) = -m$$

c-type fermions

$$m(s) = \Lambda$$

Both $\mathcal{H}_\pm$ — $s \in [-l_s, l_s]$

Only $\mathcal{H}^-$ — $s \in [-\frac{l_s}{2}, \frac{l_s}{2}]$

Only $\mathcal{H}^+$ — $s \in [-\frac{l_s}{2}, \frac{l_s}{2}]$

Boundary states

$|-\rangle$ for s=- l_s

$|+\rangle$ for s=+ l_s

$|-\rangle$ for s= $\pm\frac{l_s}{2}$

$|+\rangle$ for s= $\pm\frac{l_s}{2}$

$$e^{W_a(A_\mu)} = e^{l_s(\Lambda_+ + \Lambda_-)}\langle -|+\rangle; \quad e^{W_b(A_\mu)} = e^{l_s\Lambda_-}; \quad e^{W_c(A_\mu)} = e^{l_s\Lambda_+}$$

$$e^{W_+(A_\mu)} = \lim_{l_s\to\infty} \frac{e^{W_a(A_\mu)}}{e^{W_b(A_\mu)}e^{W_b(A_\mu)}} = \langle -|+\rangle$$

overlap formula

Some remarks

- We divided by the result for b and c type fermions. b and c have the opposite statistics compared to a.
- b and c type fermions regulate the theory by removing the contributions for all the non-zero modes of $M^\dagger M$
- We can take $\Lambda \to \infty$
- We can set H^- to $H_w = \sigma_3(\not{D}_w - m)$

Assume that the Hamiltonians are 2n by 2n matrices

$\psi_k^{\pm}, k = 1, \cdots, n$ are positive eigenstates of $H^{\pm}$ respectively

$$\langle -|+\rangle = \det O; \quad O_{jk} = [\psi_j^+]^\dagger \psi_k^- .$$

$H_+ = \sigma_3$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the positive eigenstates

$$H_w X = X\Lambda, \quad X = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}; \quad \Lambda = diag(\lambda_1^+, \cdots, \lambda_n^+, -\lambda_1^-, \cdots, -\lambda_n^-),$$

$\lambda_i^{\pm} > 0$ for all i

$$\langle -|+\rangle = \det \alpha$$

Gauge transformation

$$V(x,y) = e^{i\chi(x)}\delta(x-y) \quad H_w \to H_w^g = V^\dagger H_w V \quad X^g = \begin{pmatrix} V^\dagger\alpha & V^\dagger\gamma \\ V^\dagger\beta & V^\dagger\delta \end{pmatrix}$$

$$\int \chi(x) d^2x = 0$$

$$\langle -|+\rangle = \langle -|+\rangle^g$$

Phase ambiguity

- Phase of the eigenstates of H_w are ambiguous
- Phase can depend on the gauge field
- Only absolute value has been properly defined
- Like in continuum Pauli-Villars
  - Real part was well regulated
  - Imaginary part was cancelled for anomaly free theories
- Overlap formula was written down for any theory
- Anomaly has to show up as ambiguity in the phase
- Perturbation theory
  - Can fix the free eigenvectors
  - Upon perturbation, the perturbated eigenvector's overlap with the unperturbed one cannot be fixed.
  - Use standard Wigner-Brillouin choice -- Set the overlap between perturbed and unperturbed state as real and positive

Wigner-Brillouin phase choice

$$e^{W_+^{\mathrm{WB}}(A_\mu)} = \frac{{}_0\langle -|-\rangle}{|{}_0\langle -|-\rangle|}\langle -|+\rangle$$

One choice but not unique

Not necessarily gauge invariant

- Need to understand phase ambiguity
- Define currents as variation of the effective action
  - consistent current
- Can define another current that is unabiguous
  - covariant current
  - will be gauge covariant
  - will have anomalies

Gauge field topology

$|+\rangle$ state is always half filled

H_w need not be half filled

$$H_w = \begin{pmatrix} B-m & C \\ C^\dagger & -B+m \end{pmatrix}$$

$$H_w \begin{pmatrix} u \\ v \end{pmatrix} = 0; \quad u^\dagger u + v^\dagger v = 1,$$

$$u^\dagger B u + v^\dagger B v = m,$$

$$(B-m)u + Cv = 0; \quad C^\dagger u - (B-m)v = 0.$$

Consider a smooth deformation of $U_\mu = 1$ to U_μ^Q

There must be a zero eigenvalue somewhere in the path

If the number of particles filling the two states defining the overlap are not the same, overlap is zero.

A fermion creation or annihilation operator has to be inserted between the two states for the overlap to be non-zero.

Fermionic index $Q_f = \frac{1}{2}(n_+ - n_-)$

Right handed fermions

$$M = \partial_s + m(s); \quad m(s) = \begin{cases} -m & \text{for } s > 0 \\ \Lambda & \text{for } s \le 0 \end{cases}; \quad m, \Lambda > 0.$$

$$\langle +|-\rangle = \det \alpha^\dagger.$$

Overlap for a vector like theory has no phase ambiguity

Need to have separate set of creation and annihilation operators for left and right handed fermions

$a_{R,L}$ and $a^\dagger_{RL}$ are two sets of annihilation and creation operators

$b_{R,L} = X^\dagger a_{R,L}$ also obey canonical anticommutation relations

$$b_{R,L} = \begin{pmatrix} u'_{R,L} \\ d'_{R,L} \end{pmatrix}; \quad a_{R,L} = \begin{pmatrix} u_{R,L} \\ d_{R,L} \end{pmatrix}$$

$$|-\rangle_{R,L} = u'^\dagger_{R,L_n} u'^\dagger_{R,L_{n-1}} \cdots u'^\dagger_{R,L_2} u'^\dagger_{R,L_1} |0\rangle$$

$$|+\rangle_{R,L} = u^\dagger_{R,L_n} u^\dagger_{R,L_{n-1}} \cdots u^\dagger_{R,L_2} u^\dagger_{R,L_1} |0\rangle$$

Generating functionals

$$Z_L(\bar{\xi}_L, \xi_L) = {}_L\langle -| e^{\xi_L d_L^\dagger + \bar{\xi}_L u_L} |+\rangle_L$$

$$Z_R(\bar{\xi}_R, \xi_R) = {}_R\langle +| e^{\xi_R u_R^\dagger + \bar{\xi}_R d_R} |-\rangle_R$$

- Sources are grassmanian
- Ordering does not matter since the two terms in the exponential commute.
- $d_L^\dagger$ and u_L do not propagate since $d_L|+\rangle$ and $u_L^\dagger|+\rangle$ are zero.
- Converse holds for the other chirality.

Invariant under global chiral transformations

$$\xi_R \to e^{i\varphi_R}\xi_R; \quad \bar{\xi}_R \to \bar{\xi}_R e^{-i\varphi_R}; \quad \xi_L \to e^{i\varphi_L}\xi_L; \quad \bar{\xi}_L \to \bar{\xi}_L e^{-i\varphi_L}$$

Chiral propagators

$$G_L^{ij} = \frac{{}_L\langle -|d_{Lj}^\dagger u_{Li}|+\rangle_L}{{}_L\langle -|+\rangle_L}; \quad G_R^{ij} = \frac{{}_R\langle +|u_{Rj}^\dagger d_{LR}|-\rangle_R}{{}_L\langle +|-\rangle_R}$$

$$G_R^\dagger = G_L.$$

Left handed fermions

$$Z_L(\bar{\xi}_L, \xi_L) = \left[e^{\bar{\xi}_L [\beta\alpha^{-1}]^\dagger \xi_L} \det \alpha \right]$$

Right handed fermions

$$Z_R(\bar{\xi}_R, \xi_R) = \left[e^{\bar{\xi}_R \beta\alpha^{-1} \xi_R} \det \alpha^\dagger \right]$$

$$\xi_L d_L^\dagger + \bar{\xi}_L u_L = Q_L^- + Q_L^+$$

$$Q_L^+ = \bar{\xi}_L (\alpha^{-1})^\dagger u_L' + \xi_L d_L'^\dagger \delta^{-1}; \quad Q_L^- = -\bar{\xi}_L (\beta\alpha^{-1})^\dagger d_L - \xi_L u_L^\dagger \gamma \delta^{-1},$$

$$Q_L^+ = \xi_L d_L^\dagger + \bar{\xi}_L u_L - Q_L^-$$

$$[Q_L^+, Q_L^-] = -\bar{\xi}_L \left((\beta\alpha^{-1})^\dagger - \gamma\delta^{-1} \right) \xi_L,$$

$$e^{\xi_L d_L^\dagger + \bar{\xi}_L u_L} = e^{Q_L^+} e^{Q_L^-} e^{\frac{1}{2}\bar{\xi}_L \left((\beta\alpha^{-1})^\dagger - \gamma\delta^{-1} \right) \xi_L}.$$

$${}_L\langle - | e^{Q_L^-} = {}_L\langle - |; \quad e^{Q_L^-} | + \rangle_L = | + \rangle_L,$$

$${}_L\langle - | e^{\xi_L d^\dagger + \bar{\xi}_L u} | + \rangle_L = e^{\frac{1}{2}\bar{\xi}_L \left((\beta\alpha^{-1})^\dagger - \gamma\delta^{-1} \right) \xi_L} {}_L\langle - | + \rangle_L = e^{\bar{\xi}_L (\beta\alpha^{-1})^\dagger \xi_L} \det \alpha.$$

Overlap Dirac operator and propagator for vector gauge theories

Unitary matrix $V = \sigma_3 \epsilon(H_w).$

$$\frac{1+V}{2}X = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}; \quad \frac{1-V}{2}X = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix},$$

$$\frac{1-V}{1+V} = \begin{pmatrix} 0 & -\left(\beta\alpha^{-1}\right)^{\dagger} \\ \beta\alpha^{-1} & 0 \end{pmatrix}.$$

$$\det X = \frac{\det \alpha}{\det \delta^{\dagger}};$$

$$\det \alpha \det \alpha^{\dagger} = \det \delta \det \delta^{\dagger},$$

$$\det \frac{1+V}{2} = \det \alpha \det \alpha^{\dagger}.$$

Matrix in determiant is not same as the one in propagator

Chiral anomalies

$$U_\mu(x) \Longleftrightarrow \xi_\alpha$$

Consistent current $$j^{\rm cons}(\xi) = \partial_\alpha W_+(\xi) = \frac{\langle \partial_\alpha - | + \rangle}{\langle - | + \rangle} d\xi_\alpha$$ ambiguous

exact $$dj^{\rm cons} = 0$$

$h_\mu = 0$

$Q = 0$

$$\begin{aligned} j^{\rm cons} &= \frac{\partial \ln\langle - | + \rangle}{\partial A_\mu(x)} d\left[\partial_\mu \chi(x)\right] + \frac{\partial \ln\langle - | + \rangle}{\partial A_\mu(x)} d\left[\epsilon_{\nu\mu}\partial_\nu \phi(x)\right] \\ &= -\partial_\mu \left[\frac{\partial \ln\langle - | + \rangle}{A_\mu(x)}\right] d\chi(x) - \epsilon_{\nu\mu}\partial_\nu \left[\frac{\partial \ln\langle - | + \rangle}{A_\mu(x)}\right] d\phi(x). \end{aligned}$$

Splitting the current $$j^{\rm cons}(\xi) = j^{\rm cov}(\xi) + \Delta j(\xi)$$

$j^{\rm cov}_\alpha(\xi)$ Transformly covariantly, unabiguous, not exact

$\Delta j(\xi)$ $$d\Delta j = \left(\partial_\alpha \Delta j_\beta - \partial_\beta \Delta j_\alpha\right) d\xi_\alpha d\xi_\beta = \mathcal{F}_{\alpha\beta} d\xi_\alpha d\xi_\beta,$$ unambiguous

$d\Delta j = 0,$ $\Longrightarrow$ No anomalies

Anomaly computation on the lattice

Pick a continuum gauge field configuration

Construct the fields on the lattice

Form H_w

Compute $d\Delta j$

Take the continuum limit

If it is zero, there is no anomaly

Compute the covariant anomaly

The result will be zero in the continuum if there is no anomaly

Not necessarily zero on the lattice

Unambiguous but depends on choice of H_w

Open problem: Can one cancel anomalies exactly on the lattice?

Covariant current

$$|\partial_\alpha -\rangle = |\partial_\alpha -\rangle_\perp + |-\rangle\langle -|\partial_\alpha -\rangle.$$

$$\langle -|\partial_\alpha -\rangle_\perp = 0$$

$$j_\alpha^{\mathrm{cov}} = \frac{{}_\perp\langle \partial_\alpha - |+\rangle}{\langle -|+\rangle},$$

is unambiguous and will transform covariantly

$$\Delta j_\alpha = \langle \partial_\alpha - |-\rangle,$$

$$U_1(n_1,n_2) \to U_1'(n_1,n_2) = g^\dagger(n_1,n_2)U_1(n_1,n_2)g(n_1+1,n_2);$$
$$U_2(n_1,n_2) \to U_2'(n_1,n_2) = g^\dagger(n_1,n_2)U_2(n_1,n_2)g(n_1,n_2+1),$$

$$\delta\xi_\alpha^g = \delta\xi_\beta \left[\mathcal{D}^{-1}(g)\right]_{\beta\alpha}.$$

$$\mathcal{H}^-(\xi^g) = G^\dagger(g)\mathcal{H}^-(\xi)G(g),$$

$$\mathcal{H}^-(\xi + \delta\xi) - \mathcal{H}^-(\xi) = \delta\xi_\alpha R_\alpha(\xi).$$

$$\mathcal{H}^-(\xi^g + \delta\xi^g) - \mathcal{H}^-(\xi^g) = \delta\xi^g_\alpha R_\alpha(\xi^g) = \delta\xi_\beta \left[\mathcal{D}^{-1}(g)\right]_{\beta\alpha} R_\alpha(\xi^g),$$

$$\mathcal{H}^-(\xi^g+\delta\xi^g)-\mathcal{H}^-(\xi^g) = G^\dagger(g)\left[\mathcal{H}^-(\xi + \delta\xi) - \mathcal{H}^-(\xi)\right] G(g) = \delta\xi_\beta G^\dagger(g) R_\beta(\xi) G(g).$$

$$R_\alpha(\xi^g) = [\mathcal{D}(g)]_{\alpha\beta}\, G^\dagger(g) R_\beta(\xi) G(g).$$

$$|\partial_\alpha -\rangle_\perp(\xi) = \frac{1}{\mathcal{H}^-(\xi) - E_0(\xi)} \left[\langle -|R_\alpha(\xi)|-\rangle - R_\alpha(\xi)\right] |-\rangle; \quad \mathcal{H}^-(\xi)|-\rangle = E_0(\xi)|-\rangle.$$

$$|-\rangle^g = e^{i\phi(\xi,g)} G^\dagger(g)|-\rangle,$$

$$G(g)|+\rangle = |+\rangle$$

$$|\partial_\alpha -\rangle_\perp^g(\xi) = e^{i\phi(\xi,g)} \mathcal{D}_{\alpha\beta}(g) G^\dagger(g) |\partial_\beta -\rangle_\perp(\xi),$$

$$[j_\alpha^{\mathrm{cov}}]^g = \left[j_\beta^{\mathrm{cov}}\right] \left[\mathcal{D}^{-1}(g)\right]_{\beta\alpha},$$

Transforms covariantly and is not ambiguous

Berry's curvature

$$\mathcal{F}_{\alpha\beta} = \langle \partial_\alpha - | \partial_\beta - \rangle - \langle \partial_\beta - | \partial_\alpha - \rangle.$$

$$P = |-\rangle\langle -|; \quad P^2 = P,$$

$$\mathcal{F}_{\alpha\beta} = \mathrm{Tr}\left[(\partial_\beta P) P (\partial_\alpha P) - (\partial_\alpha P) P (\partial_\beta P) \right],$$

unambiguous

Toron background

$\phi = \chi = 0$ $\quad Q = 0.$ $\quad U_\mu(n_1, n_2) = e^{i\frac{\pi h_\mu}{L}}; \quad \mu = 1, 2$

eigenvectors $\frac{1}{\sqrt{2\mu(\mu-\alpha)}}\begin{pmatrix} \beta & \mu-\alpha \\ \mu-\alpha & -\beta^* \end{pmatrix}$ eigenvalues μ and $-\mu$

$$
\begin{aligned}
\mu^2 &= \alpha^2 + \beta\beta^*; \\
\alpha &= 2\sum_\mu \sin^2\left[\frac{\pi}{L}\left(p_\mu + \frac{h_\mu}{2}\right)\right] - m; \\
\beta &= i\left\{\sin\left[\frac{2\pi}{L}\left(p_1 + \frac{h_1}{2}\right)\right] - i\sin\left[\frac{2\pi}{L}\left(p_2 + \frac{h_2}{2}\right)\right]\right\}; \\
&-\frac{L}{2} \le p_\mu < \frac{L}{2}.
\end{aligned}
$$

$$e^{W_-(h_\mu)} = \prod_{p_\mu} \frac{\beta}{\sqrt{2\mu(\mu-\alpha)}}$$

Role of m in the overlap

$m > 0$ $L \to \infty$

$p_\mu \approx 0$

$$\alpha \approx 2\frac{\pi^2}{L^2}\sum_\mu \left(p_\mu + \frac{h_\mu}{2}\right)^2 - m; \quad \beta \approx i\frac{2\pi}{L}\left\{\left(p_1 + \frac{h_1}{2}\right) - i\left(p_2 + \frac{h_2}{2}\right)\right\}.$$

$\sqrt{2\mu(\mu-\alpha)}$ ~ $2m$

factor in the determinant $\left(p_1 + \frac{h_1}{2}\right) + i\left(p_2 + \frac{h_2}{2}\right)$

$p_1 \approx \frac{L}{2}$

$p_2 \approx 0$

$$\alpha \approx 2\frac{\pi^2}{L^2}\sum_\mu \left(p_\mu + \frac{h_\mu}{2}\right)^2 + 2 - m; \quad \beta \approx -i\frac{2\pi}{L}\left\{\left(p_1 + \frac{h_1}{2}\right) + i\left(p_2 + \frac{h_2}{2}\right)\right\}$$

If $0 < m < 2$ then $\mu \approx (2-m)\left[1 + \frac{3-m}{2(2-m)^2}\frac{4\pi^2}{L^2}\sum_\mu\left(p_\mu + \frac{h_\mu}{2}\right)^2\right]$

$\sqrt{\mu(\mu-\alpha)}$ ~ $\frac{2\pi}{L}\sqrt{\sum_\mu\left(p_\mu + \frac{h_\mu}{2}\right)^2}$

Contributes a factor of order one to the determinant

Chiral anomaly

Phase is ambiguous

Wigner Brillouin phase choice
continuum limit

$$e^{W_-^{\mathrm{WB}}(h_\mu)} = e^{\frac{\pi}{8}h(h-\bar{h})}\frac{\vartheta(\frac{h}{2};i)}{\eta(i)}; \quad h = h_1 + ih_2,$$

$$P = |-\rangle\langle-|$$

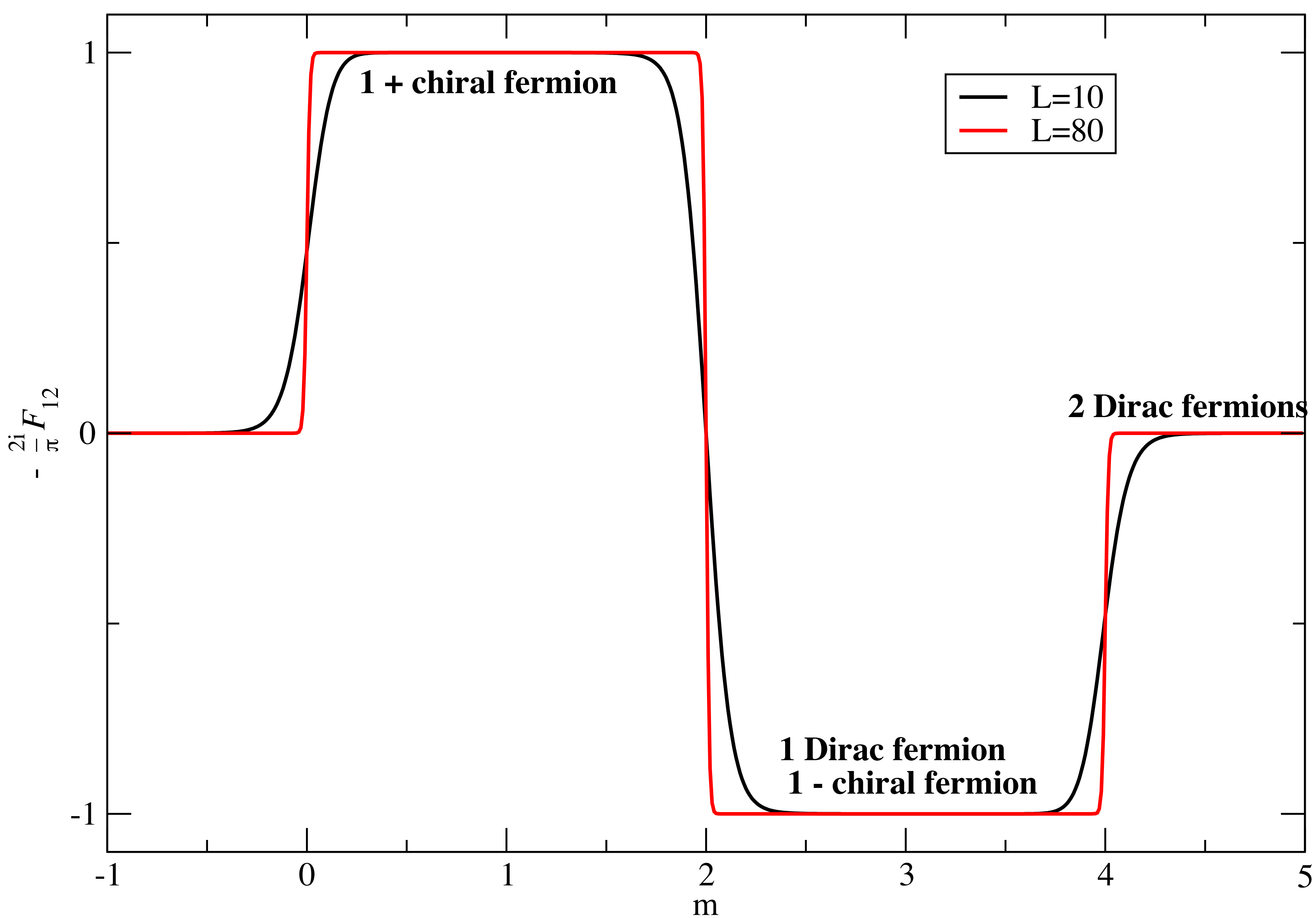
$$\mathcal{F}_{\alpha\beta} = \mathrm{Tr}\left[(\partial_\beta P)P(\partial_\alpha P) - (\partial_\alpha P)P(\partial_\beta P)\right]$$

$$P_p(h) = \frac{1}{2} - \frac{1}{2\mu}\left[\alpha\sigma_3 + \beta_1\sigma_1 - \beta_2\sigma_2\right] = \frac{1}{2} + \frac{1}{2}\hat{w}_p(h)\cdot\vec{\sigma}$$

$$\beta_\mu = \frac{2\pi}{L}\left((p_\mu + \frac{h_\mu}{2}\right)$$

$$\mathcal{F}_{12}(h) = \frac{i}{2}\sum_p \vec{w}_p \cdot \left(\frac{\partial\vec{w}_p}{\partial h_1} \wedge \frac{\partial\vec{w}_p}{\partial h_2}\right)$$

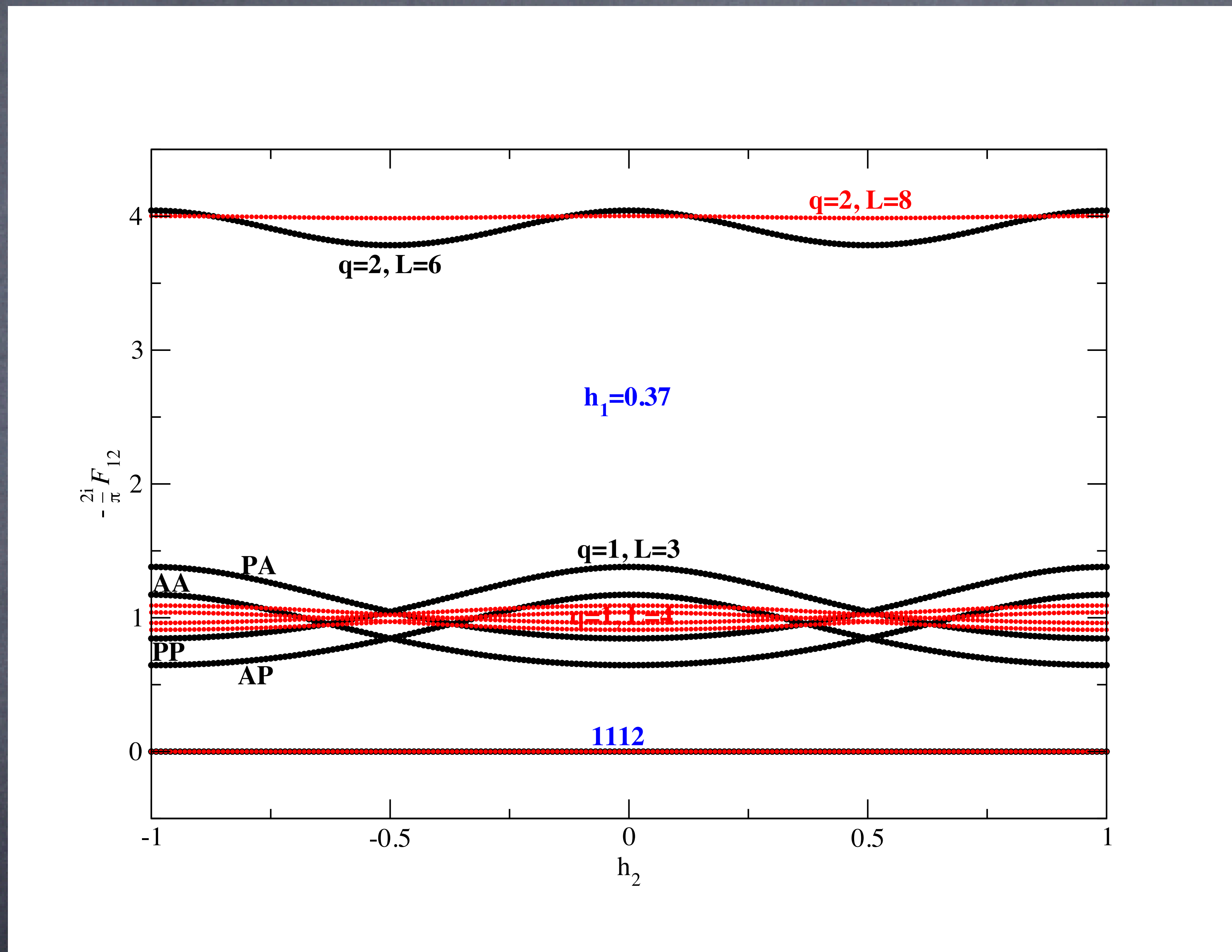
- sum becomes an integral in the continuum limit
- independent of toron variables since we can shift the integral using periodic boundary conditions.
- integrand is an area element.
- integral counts the winding on the map of torus onto the sphere.
- One winding for 0 < m < 2.


1 + chiral fermion
L=10
L=80
2 Dirac fermions
1 Dirac fermion
1 - chiral fermion
$-\frac{2i}{\pi}F_{12}$
m
1
0
-1
-1
0
1
2
3
4
5

Berry's curvature is proportional to the square of the charge

Results related by complex conjugate for + and - chirality

11112 model - 4 q=1 with + chirality and 1 q=2 with - chirality



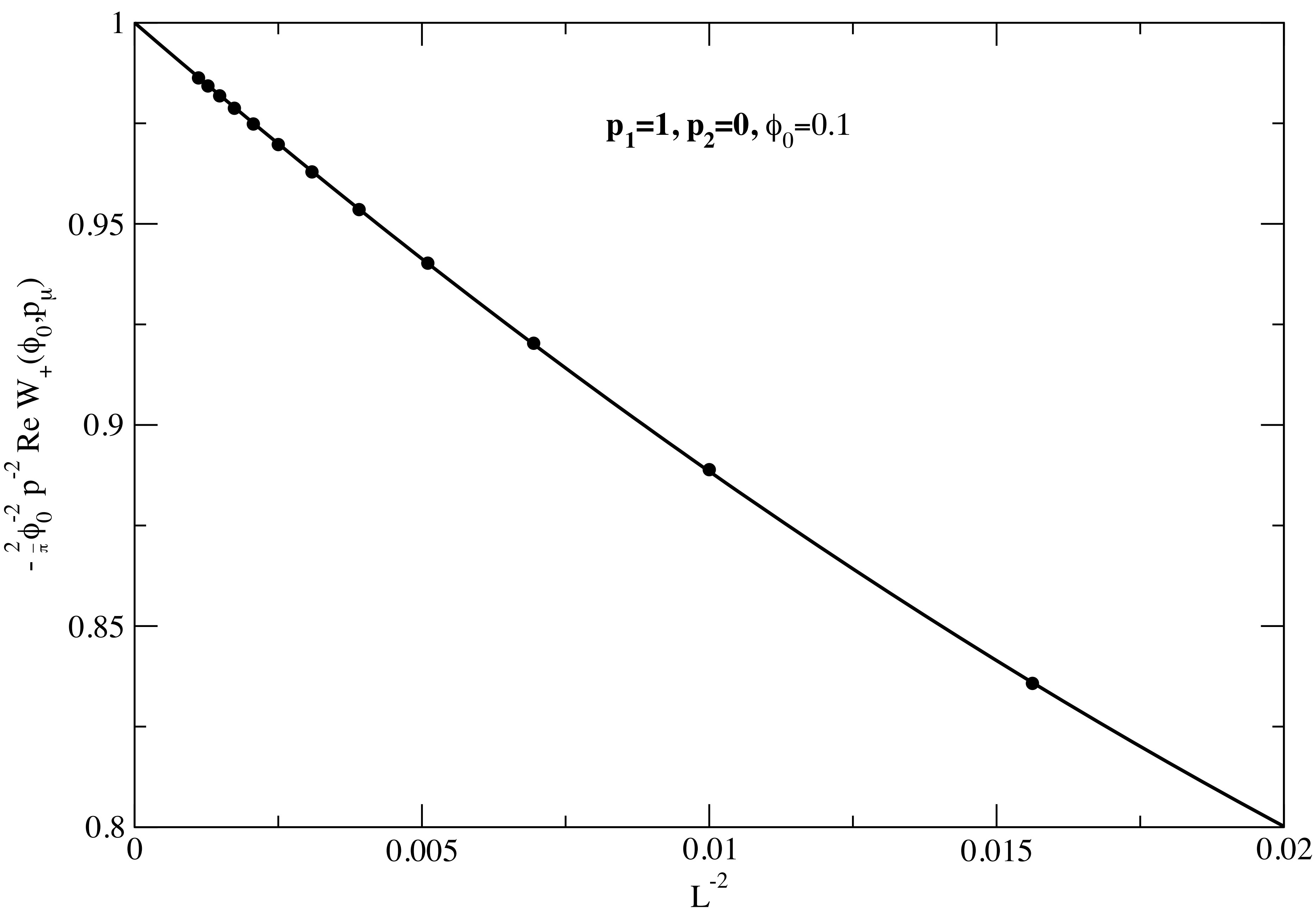
Delicate cancellation due to a theta function identity

Consistent and covariant anomalies
A numerical computation

$$W_-(A_\mu) = \sum_p \left[-\phi(p)\phi(-p)f(p) + i\phi(p)\chi(-p)g(p)\right].$$

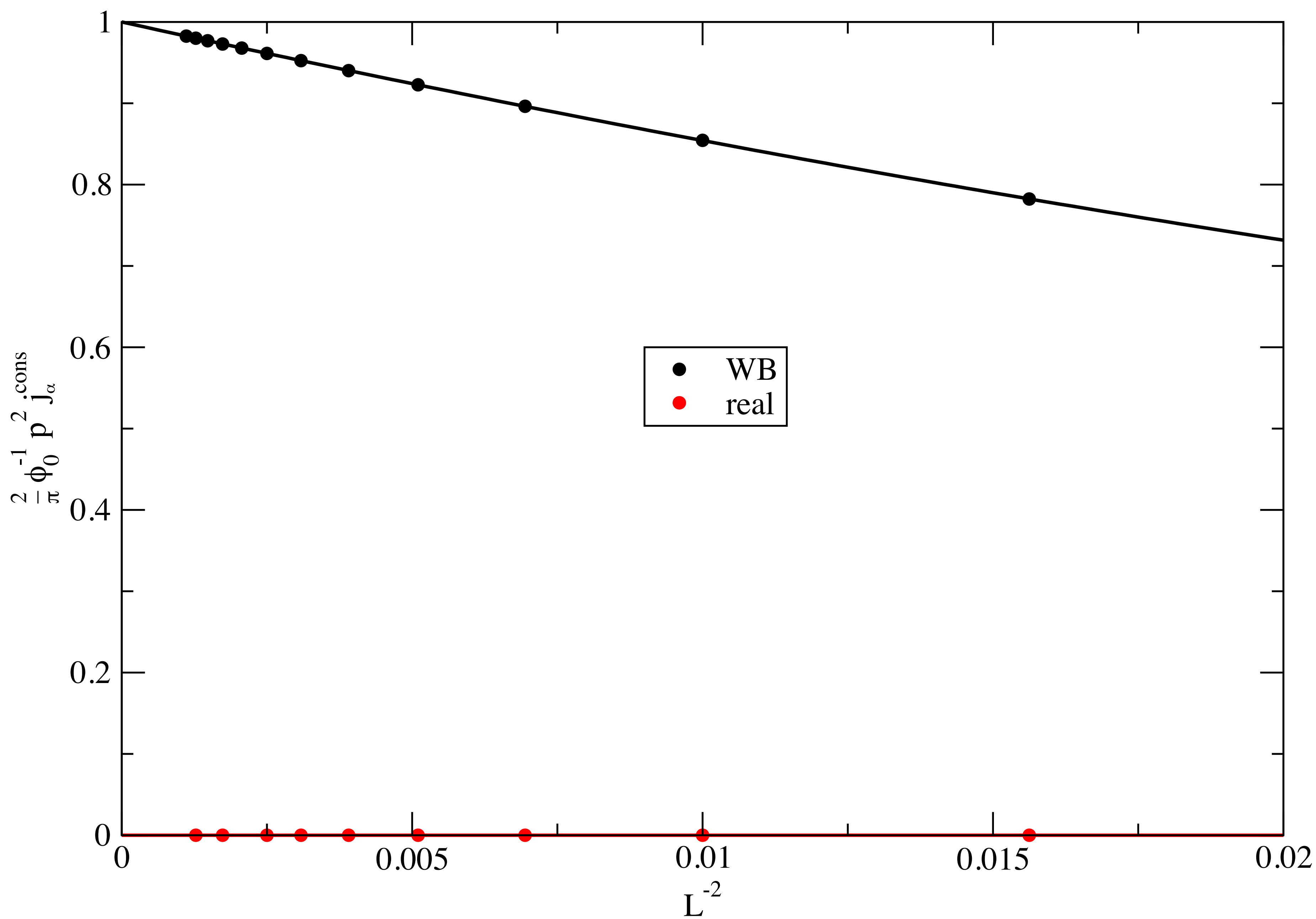
$$\phi(n_1, n_2) = \phi_0 \cos\left[\frac{2\pi}{L}(p_1 n_1 + p_2 n_2)\right]$$

Pick gauge transformation to have the same momentum

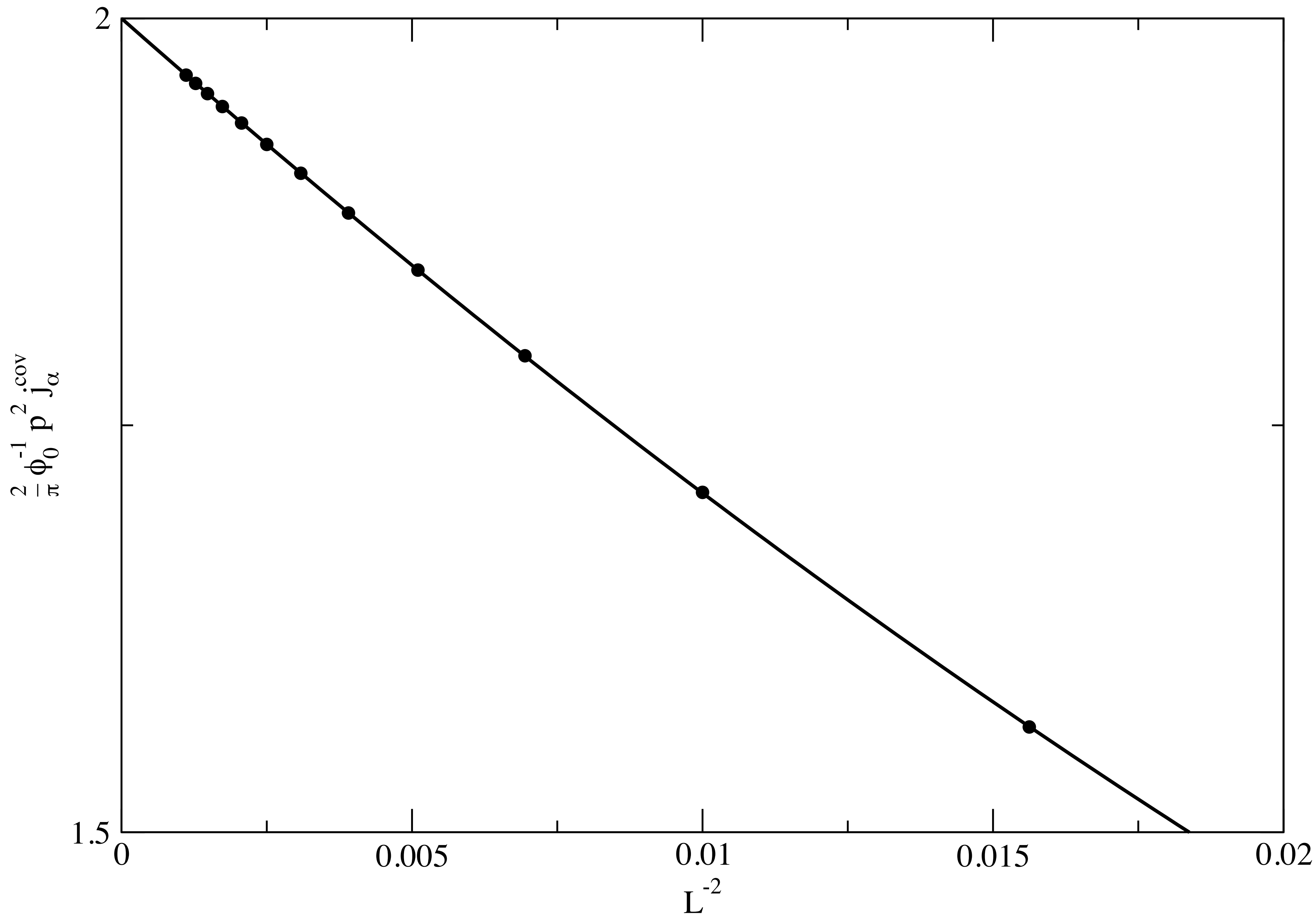


p_1=1, p_2=0, ϕ_0=0.1
$-\frac{2}{\pi}\phi_0^{-2}\,p^{-2}\,\mathrm{Re}\,W_+(\phi_0;p_\mu)$
L^{-2}
1
0.95
0.9
0.85
0.8
0
0.005
0.01
0.015
0.02

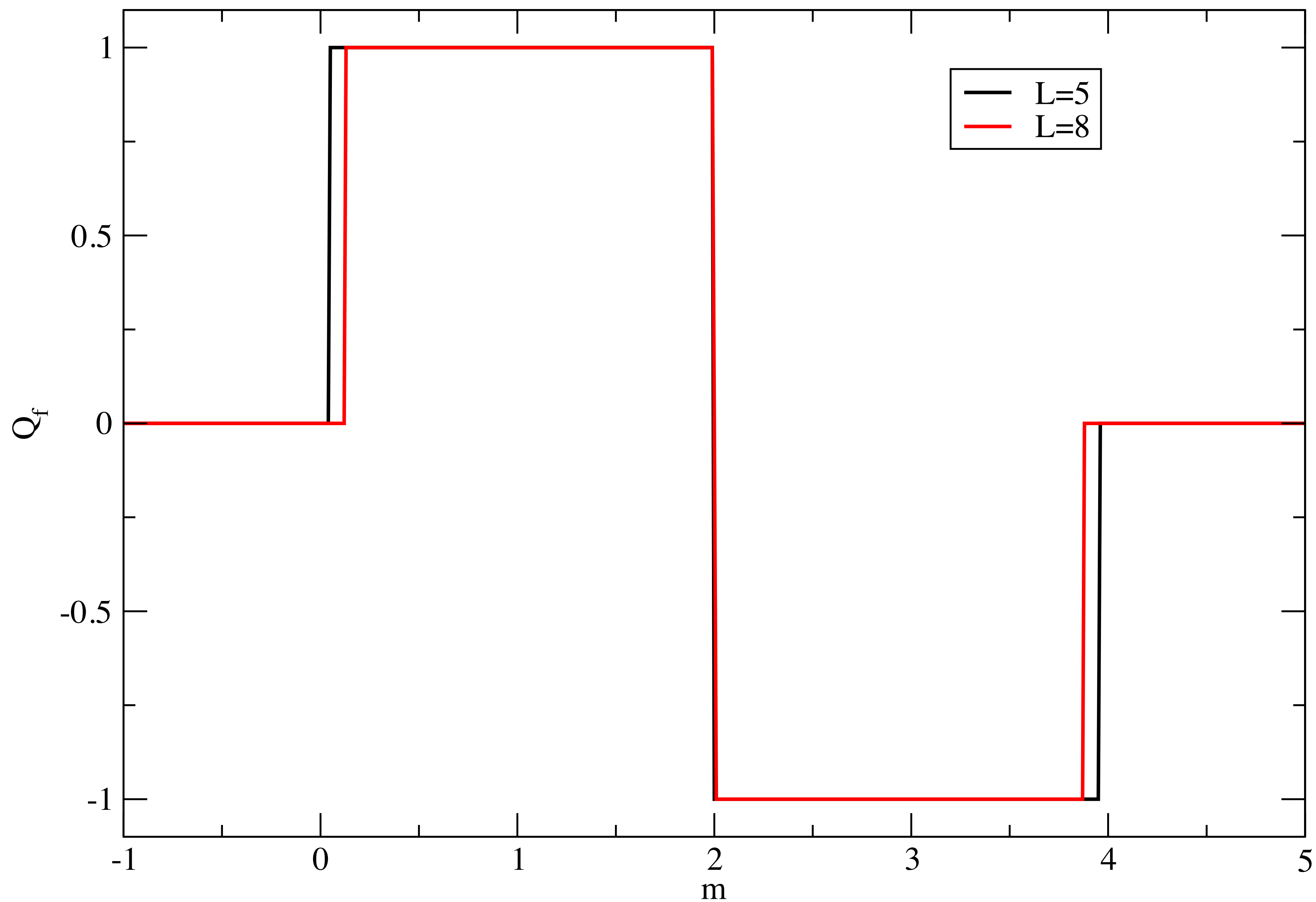
Consistent anomaly



Covariant anomaly



Fermionic index versus m



Fermionic index versus Q