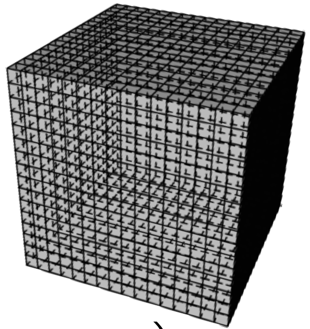
2. phase transition and critical scaling

Critical phenomenon

Phase transition: a drastic change of system properties
<= **singularity** of the partition fn. (free en.) at the trans. pt.



Lattice with finite extent:
mathematically well-defined analytic system.

Non-analyticities appear only in the thermodyn. lim. ($N_s \to \infty$), i.e.
singularities caused by long-range modes (massless modes).

*We may describe the singular properties around a phase trans. by an **effective theory of long-range modes**.*

Long-range modes and G-L theory

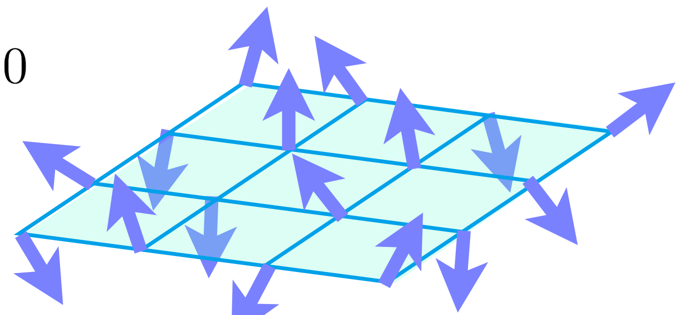
[example] 2nd order transition of a ferromagnetic spin system

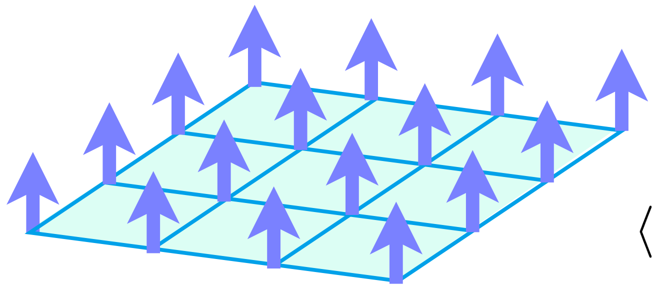
high T: disordered

Tc

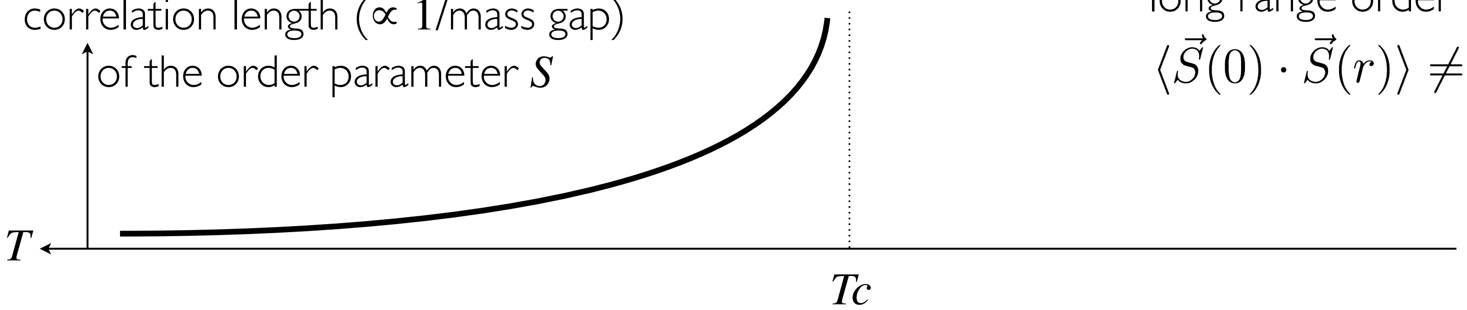
low T: ordered

$\langle \vec{S} \rangle = 0$





$\langle \vec{S} \rangle \neq 0$



long range order

$\langle \vec{S}(0) \cdot \vec{S}(r) \rangle \neq 0 \qquad r \to \infty$

The order parameter carries the d.o.f. for the long-range mode.

The critical properties near the 2nd order transition

<= *effective theory of the order param.:* **Ginsburg-Landau theory**

How to extract the long-range modes

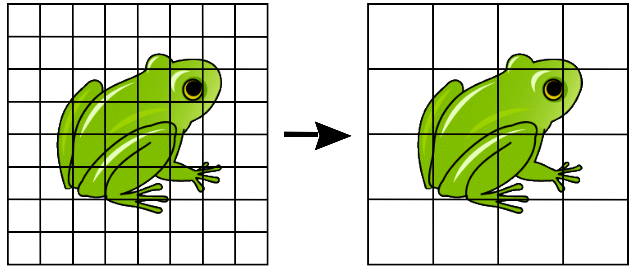
Integrate out the short range modes below a scale bxa.

$$Z = \int \mathcal{D}\phi \, e^{-S(\phi)} = \int \mathcal{D}\phi' \, e^{-S'(\phi')}, \qquad \phi' = \phi(\lambda > ba)$$

$$a \longrightarrow a' = ba$$

$$\xi \longrightarrow \xi' = \xi/b$$

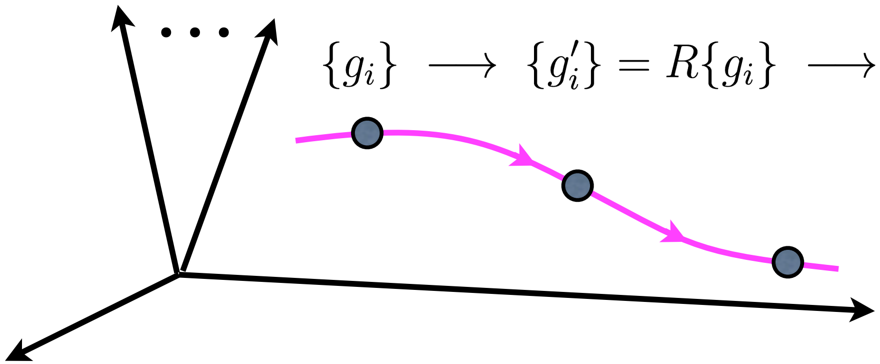
corr. length
in units of a (a').



This block transformation defines a flow (**RG flow**) in the infinite-dimensional coupl. param. space:

$$S(\phi) = \sum g_i \, O_i(\phi) \longrightarrow S'(\phi') = \sum g_i' \, O_i(\phi')$$

$$\{g_i\} \longrightarrow \{g_i'\} = R\{g_i\} \longrightarrow \{g_i''\} = R^2\{g_i\} \longrightarrow \cdots$$



The goal of this flow is the eff. theory of the long-range modes!

RG flow and FP

Let us introduce several terms:

- **Fixed point** (FP): invariant point under the block transformation

  $\xi \longrightarrow \xi' = \xi/b$ corr. length in units of a (a').

  Thus, FP => $\xi = 0$ or ∞ (the converse is not always true).

  We are interested in long-range modes. Let us denote FP with $\xi = \infty$ as g^*.

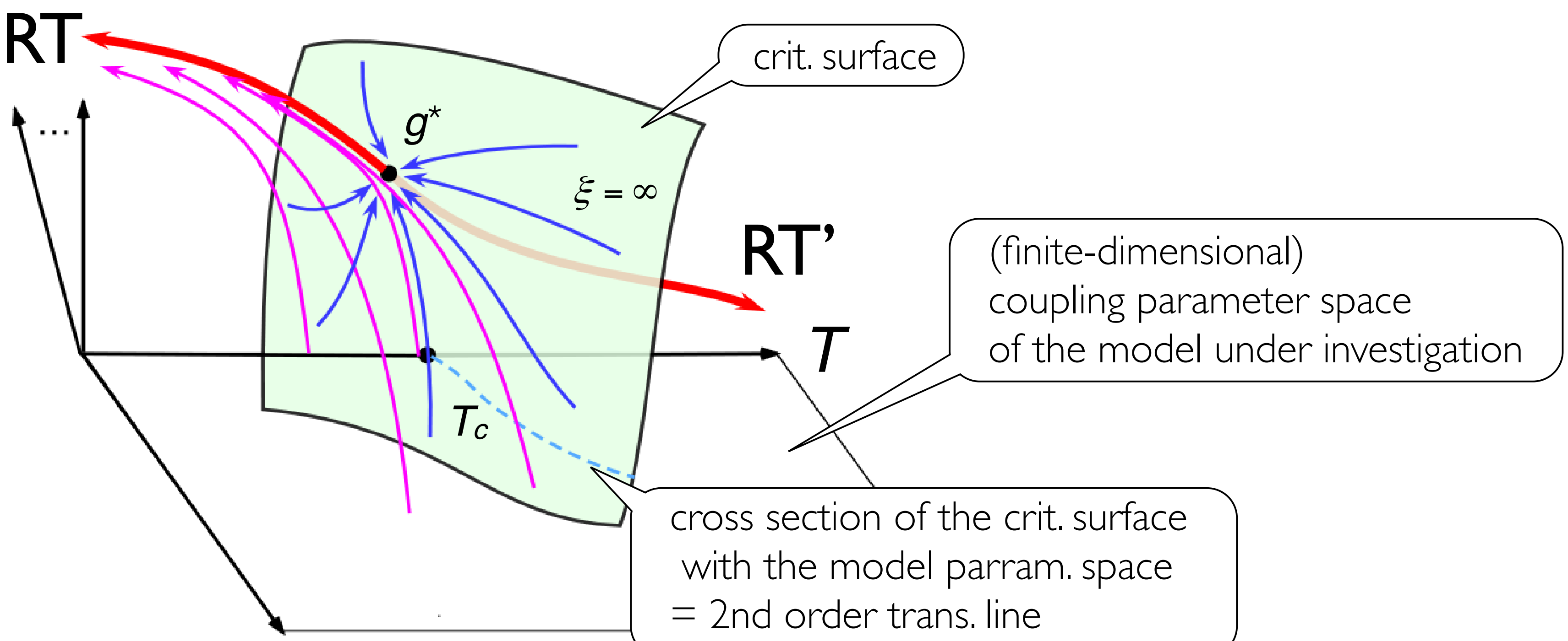
- **Critical surface**: {points} which flow into a g^*: $R^\infty\{g\} = \{g^*\}$

  $\xi = \infty$ on the crit. surface

  => 2nd order transition on the crit. surface

RG flow and FP

For simplicity, let us consider the case g^* is isolated,

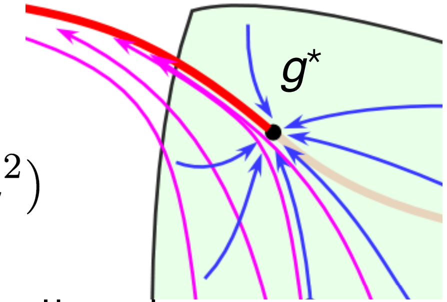


Because we are interested in the long-range modes around the transition, consider a point in the initial coupl. paaram. space *near (but not on)* the trans. line, and apply block transformations.

Due to the flow on the crit. surface, the point will first flows along the crit. surface towards g^*.

Because ξ is decreasing by the block transformations, the flow will eventually deviate from the crit. surface where $\xi = \infty$, and will flow along a line (**Renormalized Trajectory**) starting from the g^*, towards $\xi = 0$. RT reflects the properties of the IR eff. theory.

RG flow around g^*



$$g_i = g_i^* + h_i \longrightarrow g_i' = g_i^* + h_i', \quad h_i' = \sum_j B_{ij}(b) h_j + O(h^2)$$

Rearrange the basis operators O_i so that B_{ij} is diagonalized:

$$h_i' = b^{y_i} h_i$$

y_i : critical exponent / anormalous dimension

★ $y_i > 0$ => h_i -> large with block transf. "relevant" => the direction of RT

★ $y_i < 0$ => h_i -> 0 "irrelevant" => directions on the crit. surface

★ $y_i = 0$ "marginal" <= g^* is not isolated
g^* continuous in these directions

$y_i \approx -$ (dimension of O_i) + quantum corrections

For the singularities around the transition point, only "relevant" operators are relevant.

Possible variety of relevant operators <= spatial dimension, symmetry, etc.
However, the number is limited because both field and derivative increase the dim. of O_i.

Critical scaling

[example] 2nd order ferromagnetic transition of 3d spin system

relevant O_i h_i

$s^2 \sim E$ $t = (T_c - T)/T_c$

$s^1 \sim M$ h external magnetic field

=> free energy

$$F = F(t,h) + \text{less singular terms}$$

Because Z is invariant under the block transformation, F is so too up to a trivial scale factor.

$$F(t,h) \to b^{-d} F\left(b^{y_t} t, b^{y_h} h\right) = F(t,h)$$

From this equation, we obtain a chain of scaling relations.

Specific heat, magnetization, susceptibilities:

$$\begin{cases} C(t,h=0) \equiv V\left\{\langle E^2\rangle - \langle E\rangle^2\right\} = \dfrac{\partial^2}{\partial t^2} F(t,h)\Big|_{h=0} = \dfrac{\partial^2}{\partial t^2}\left[b^{-d} F\left(b^{y_t} t, b^{y_h} h\right)\right]\Big|_{h=0} \\ \qquad = b^{2y_t - d} \dfrac{\partial^2}{\partial x^2} F(x,0)\Big|_{x=b^{y_t} t} = t^{(d-2y_t)/y_t} \dfrac{\partial^2}{\partial x^2} F(x,0)\Big|_{x=1} \propto t^{-\alpha}, \quad \alpha = (d - 2y_t)/y_t \\ M(t,h=0) \propto t^{\beta}, \qquad \beta = (d - y_h)/y_t \\ \chi(t,h=0) \equiv V\left\{\langle M^2\rangle - \langle M\rangle^2\right\} \propto t^{-\gamma}, \quad \gamma = (2y_h - d)/y_t \\ M(t=0,h) \propto h^{1/\delta}, \qquad 1/\delta = (d - y_h)/y_h \end{cases}$$

Critical scaling (2)

correlation length:

$$\xi(t,h) = b\xi\left(b^{y_t}t, b^{y_h}h\right) \quad \Rightarrow \quad \xi(t,h=0) \propto t^{-\nu}, \quad \nu = 1/y_t$$

correlation function $\quad G(k;t=h=0) \propto k^{-2+\eta} \quad (k \to 0), \quad \eta = d+2-2y_h$

(hyper)-scaling relations:

$$\alpha = 2 - d\nu = 2 - \beta(\delta+1), \beta = \nu(\delta - 2 + \eta)/2, \gamma = \nu(2-\eta) = \beta(\delta-1), \delta = (d+2-\eta)/(d-2+\eta)$$

scaling function:

$$F(t,h) = b^{-d}F\left(b^{y_t}t, b^{y_h}h\right)$$

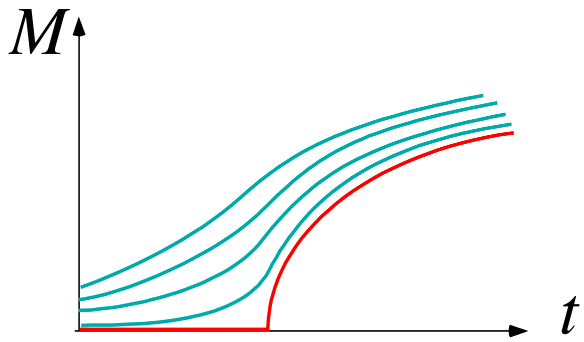
$$\Rightarrow \quad M(t,h) = b^{y_h-d}M\left(b^{y_t}t, b^{y_h}h\right) = h^{(d-y_h)/y_h}M\left(h^{-y_t/y_h}t, 1\right) = h^{1/\delta}f_M\left(t/h^{1/\beta\delta}\right)$$

$$\Rightarrow \quad \chi(t,h) = h^{(1/\delta)-1}f_\chi\left(t/h^{1/\beta\delta}\right)$$

pseudo-critical point at h ≠ 0,
defined as the peak position of the scsceptibility:

$$t_{pc} \equiv \frac{T_C - T_{pc}}{T_C} \propto h^{1/\beta\delta}$$

$$\chi_{\max} \propto h^{(1/\delta)-1}$$



etc.

Finite size scaling

We extend the scaling arguments to systems with finite volume L^d.
=> F gets a new parameter L, but the b-dep. of L is known!!

$$L \to L' = b^{-1}L$$

$$F(t,h,L) = b^{-d}F\left(b^{y_t}t, b^{y_h}h, b^{-1}L\right)$$

$$\Rightarrow \quad \chi(t,h=0,L) = b^{2y_h-d}\chi\left(b^{y_t}t,0,b^{-1}L\right) = L^{2y_h-d}\chi\left(L^{y_t}t,0,1\right)$$

$$= L^{2y_h-d}\tilde{f}_\chi\left(L^{y_t}t\right) = L^{\gamma/\nu}\tilde{f}_\chi\left(L^{1/\nu}t\right)$$

$$\therefore \quad \chi_{\max}(L) \propto L^{\gamma/\nu}, \quad T_C - T_{pc}(L) \propto L^{-1/\nu}, \quad \cdots$$

The case of <u>1st order transition</u>

Formally follows a scaling similar to the 2nd order case
<= spin wave approximation, double Gaussian approximation, etc.

$$\chi_{\max}(L) \propto V \sim L^d$$

Universality

Different points are attracted to the same g^* and then RTs from that.
In the infinite dimensional parameter space, we have model parameter spaces for other systems too.
When the points on different systems are attracted to the same g^*, they have the same ("**universal**") IR effective theory and thus the same critical properties.
Because the variety of possible effective IR theories is limited, the variety of the critical property ("**universality class**") is also limited. On the other hand, a system may have different regions in the model parameter space for different universality classes.