Numerical tools for manipulating quark propagators and correlation functions

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Quark-field smearing

Smearing - an essential ingredient for precision

- To build an operator that projects effectively onto a low-lying hadronic state need to use smearing
- Instead of the creation operator being a direct function applied to the fields in the lagrangian first smooth out the UV modes which contribute little to the IR dynamics directly.
- A popular gauge-covariant smearing algorithm; Jacobi/Wuppertal smearing: Apply the linear operator

$$\square_J = \exp(\sigma \Delta^2)$$

- Δ^2 is a lattice representation of the 3-dimensional gauge-covariant laplace operator on the source time-slice

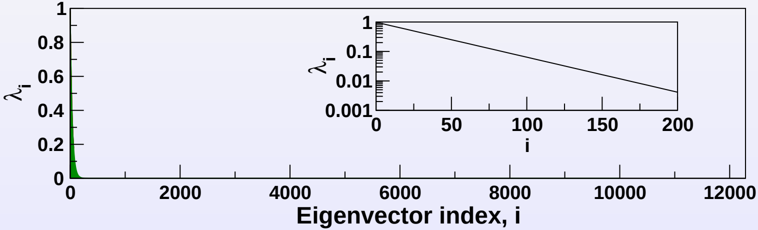
$$\Delta^2_{x,y} = 6\delta_{x,y} - \sum_{i=1}^{3} U_i(x)\delta_{x+\hat{\imath},y} + U_i^\dagger(x-\hat{\imath})\delta_{x-\hat{\imath},y}$$

- Correlation functions look like $\mathrm{Tr}\ \square_J M^{-1} \square_J M^{-1} \square_J \ldots$

- **Gaussian** smearing:

$$\lim_{n\to\infty}\left(1+\frac{\sigma\nabla^2}{n}\right)^n = \exp(\sigma\nabla^2)$$

- This acts in the space of coloured scalar fields on a time-slice: $N_s \times N_c$



- Data from $a_s \approx 0.12\text{fm}\ 16^3$ lattice: $16^3 \times 3 = 12288$.

Can redefining smearing help?

- Computing quark propagation in configuration generation and observable measurement is expensive.
- Objective: extract as much information from correlation functions as possible.

Two problems:

1. Most correlators: signal-to-noise falls exponentially
2. Making measurements can be costly:
   - Variational bases
   - Exotic states using more sophisticated creation operators
   - Isoscalar mesons
   - **Multi-hadron states**

- Good operators are **smeared**; helps with problem 1, can it help with problem 2?

- **Smeared field:** $\tilde{\psi}$ from ψ, the “raw” quark field in the path-integral:

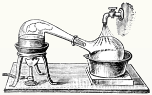
$$\tilde{\psi}(t) = \Box[U(t)]\ \ \psi(t)$$

- Extract the essential degrees-of-freedom.
- Smearing should preserve symmetries of quarks.
- Now form creation operator (e.g. a meson):

$$O_M(t) = \bar{\tilde{\psi}}(t)\Gamma\tilde{\psi}(t)$$

- Γ: operator in $\{\underline{s}, \sigma, c\} \equiv \{\text{position,spin,colour}\}$
- Smearing: overlap $\langle n|O_M|0\rangle$ is large for low-lying eigenstate $|n\rangle$

"*distill*: to extract the quintessence of" [OED]



- Distillation: **define** smearing to be explicitly a very low-rank operator. Rank is $N_{\mathcal{D}}(\ll N_s \times N_c)$.

Distillation operator

$$\Box(t) = V(t)V^{\dagger}(t)$$

with $V^{a}_{\underline{x},c}(t)$ a $N_{\mathcal{D}} \times (N_s \times N_c)$ matrix

- Example (used to date): $\Box_{\nabla}$ the **projection operator into $\mathcal{D}_{\nabla}$, the space spanned by the lowest eigenmodes of the 3-D laplacian**
- Projection operator, so idempotent: $\Box_{\nabla}^2 = \Box_{\nabla}$
- $\lim_{N_{\mathcal{D}} \to (N_s \times N_c)} \Box_{\nabla} = I$
- Eigenvectors of ∇^2 not the only choice...

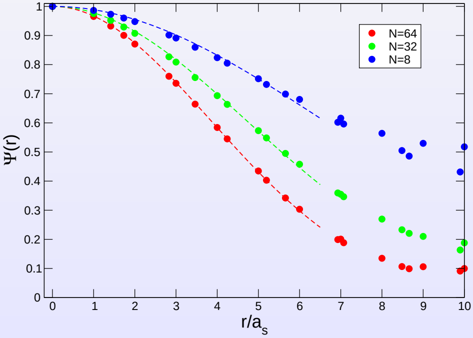
Distillation: preserve symmetries

- Using eigenmodes of the gauge-covariant laplacian **preserves lattice symmetries**

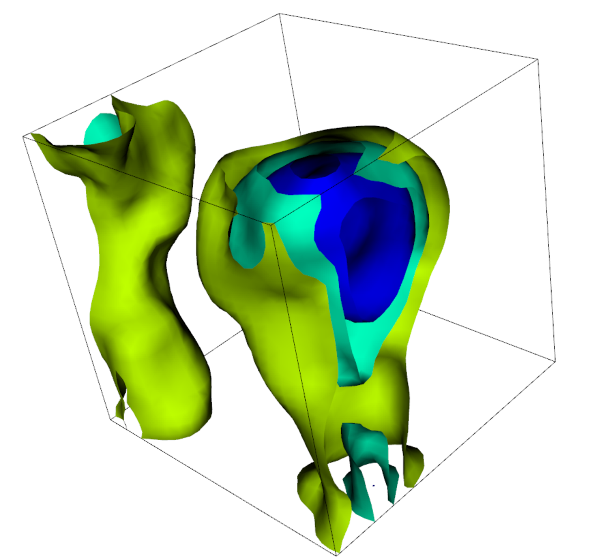
$$U_i(\underline{x}) \xrightarrow{g} U_i^g(\underline{x}) = g(\underline{x}) U_i(\underline{x}) g^\dagger(\underline{x} + \underline{\hat{\imath}})$$

$$\Box_\nabla(\underline{x}, \underline{y}) \xrightarrow{g} \Box_\nabla^g(\underline{x}, \underline{y}) = g(\underline{x}) \Box_\nabla(\underline{x}, \underline{y}) g^\dagger(\underline{y})$$

- Translation, parity, charge-conjugation symmetric
- O_h symmetric
- Close to $SO(3)$ symmetric
- "local" operator



Eigenmodes of the laplacian



- Lowest mode on a $32^3 \equiv (3.8\ \mathrm{fm})^3$ lattice.

- Consider an isovector meson two-point function:

$$C_M(t_1 - t_0) = \langle\langle \bar{u}(t_1)\Box_{t_1}\Gamma_{t_1}\Box_{t_1} d(t_1) \quad \bar{d}(t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0} u(t_0)\rangle\rangle$$

- Integrating over quark fields yields

$$C_M(t_1 - t_0) = \\ \langle \mathrm{Tr}_{\{\underline{s},\sigma,c\}} \left(\Box_{t_1}\Gamma_{t_1}\Box_{t_1} M^{-1}(t_1, t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0} M^{-1}(t_0, t_1) \right) \rangle$$

- Substituting the low-rank distillation operator $\Box$ reduces this to a **much smaller** trace:

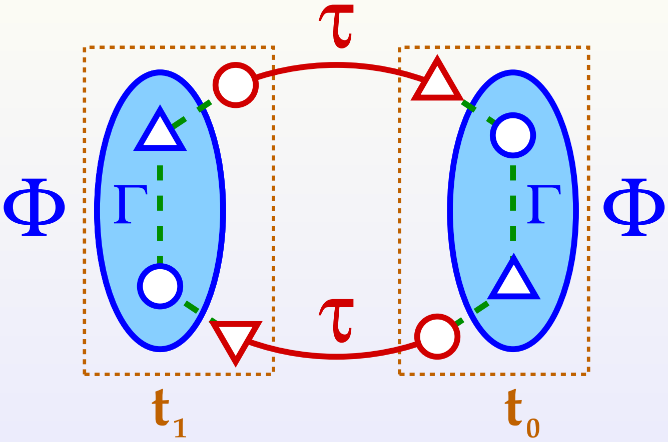
$$\boxed{C_M(t_1 - t_0) = \langle \mathrm{Tr}_{\{\sigma,\mathcal{D}\}} \left[\Phi(t_1)\tau(t_1, t_0)\Phi(t_0)\tau(t_0, t_1) \right] \rangle}$$

- $\Phi^{\alpha,a}_{\beta,b}$ and $\tau^{\alpha,a}_{\beta,b}$ are $(N_\sigma \times N_\mathcal{D}) \times (N_\sigma \times N_\mathcal{D})$ matrices.

$$\boxed{\Phi(t) = V^\dagger(t)\Gamma_t V(t)} \qquad \boxed{\tau(t, t') = V^\dagger(t) M^{-1}(t, t') V(t')}$$

The "perambulator"

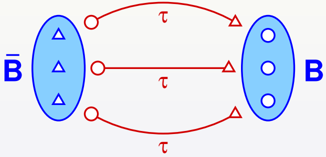
Meson two-point function



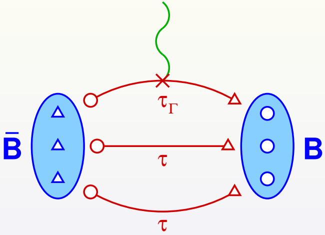
Distilled meson two-point correlation function

$$C_M(t_1 - t_0) = \mathrm{Tr}_{\{\sigma,D\}}\left[\Phi(t_1)\ \tau(t_1,t_0)\ \Phi(t_0)\ \tau(t_0,t_1)\right]$$

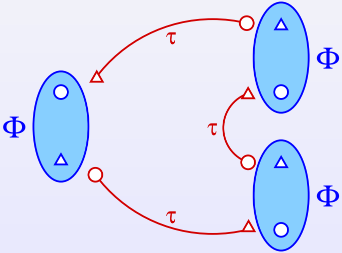
More diagrams



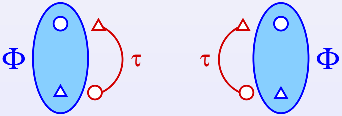
$\bar{B}_{abc}\tau_{aa'}\tau_{bb'}\tau_{cc'}B_{a'b'c'}$



$\bar{B}_{abc}\tau_{aa'}\tau^{\Gamma}{}_{bb'}\tau_{cc'}B_{a'b'c'}$

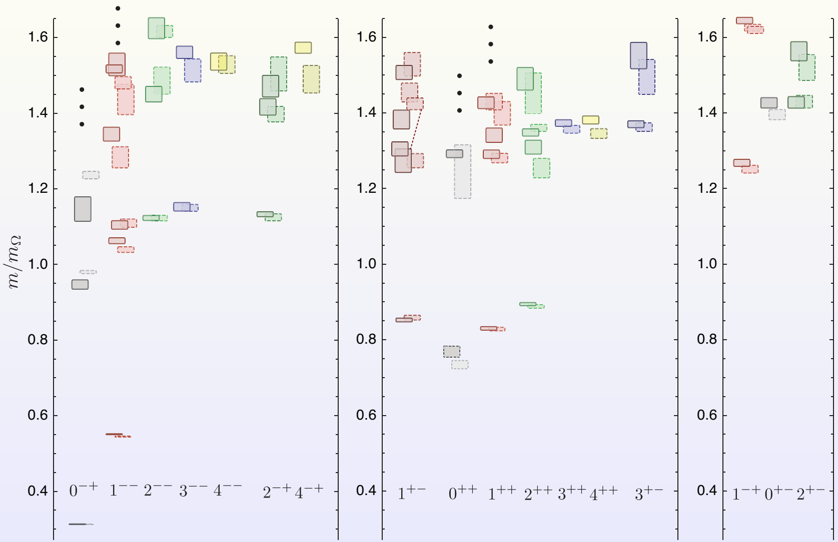


$\mathrm{Tr}\left[\Phi\tau\Phi\tau\Phi\tau\right]$



$\mathrm{Tr}\left[\Phi\tau\right]\ \mathrm{Tr}\left[\Phi\tau\right]$

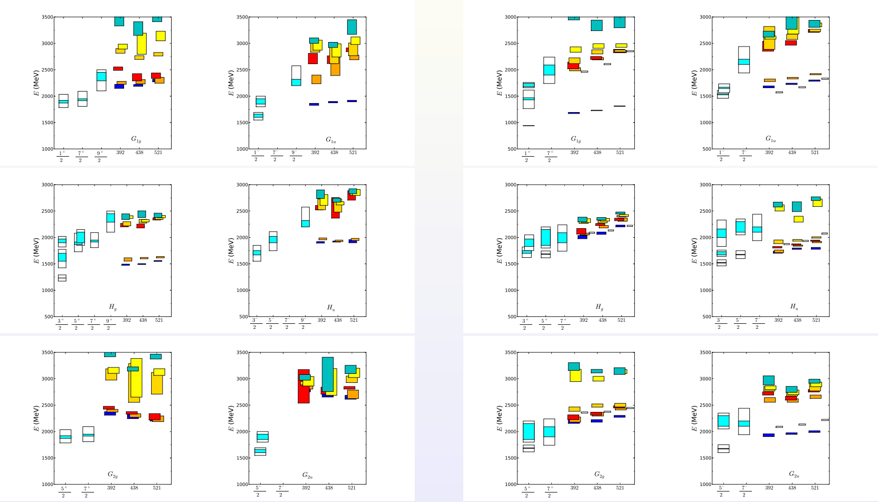
Isoscalar meson spectrum



- $16^3, 20^3$ lattice (about 2-2.5 fm), $m_\pi \approx 440$ MeV

  [arXiv:0909.0200, arXiv:1004.4930]

Baryon spectra

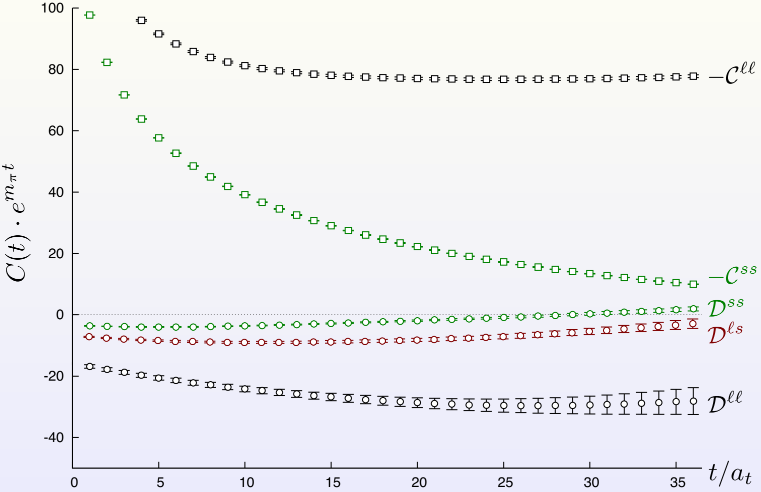


Δ N

- 16^3 lattice (about 2 fm). Three pion masses

[arXiv:1011.1509]

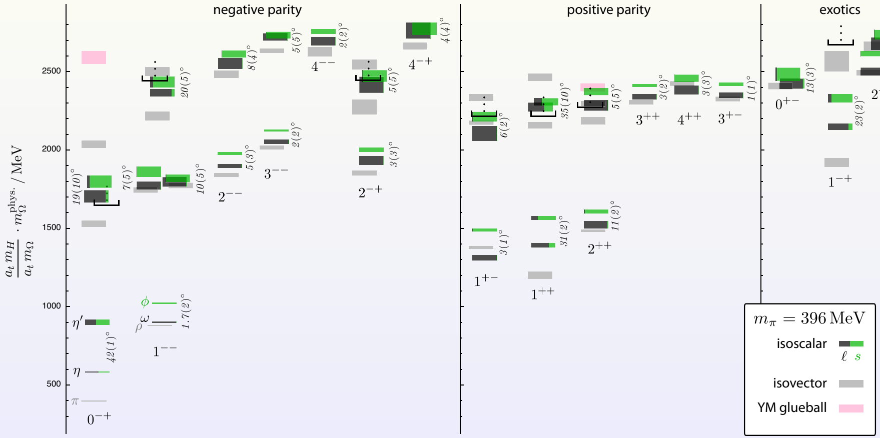
Isoscalar meson (η') correlation function



- Correlation functions for $\bar{\psi}\gamma_5\psi$ operator, with different flavour content (s, l).
- 16^3 lattice (about 2 fm).

[arXiv:1102.4299]

Isoscalar meson spectrum



- 16^3 lattice (about 2 fm), $m_\pi \approx 400$ MeV
- Black/green bars indicate flavour mixing

Bad news - the price tag

- So far - good results on modest lattice sizes $N_s = 16^3 \equiv (1.9\text{fm})^3$.
- Used $N_{\mathcal{D}} = 64$ for mesons, $N_{\mathcal{D}} = 32$ for baryons

The problem:

- To maintain constant resolution, need $N_{\mathcal{D}} \propto N_s$
- **Budget:**

Fermion solutions	construct τ	$\mathcal{O}(N_s^{\mathbf{2}})$
Operator constructions	construct Φ	$\mathcal{O}(N_s^{\mathbf{2}})$
Meson contractions	$\mathrm{Tr}[\Phi\tau\Phi\tau]$	$\mathcal{O}(N_s^{\mathbf{3}})$
Baryon contractions	$\bar{B}\tau\tau\tau B$	$\mathcal{O}(N_s^{\mathbf{4}})$

- 32^3 lattice: $64 \times (\frac{32}{16})^3 = 512$ — too expensive.
- Some benefits in reduction in variance with N_s
- **Can stochastic estimation technology help?**

Stochastic estimation in the distillation space

- Construct a **stochastic identity matrix in $\mathcal{D}$**: introduce a vector η with $N_{\mathcal{D}}$ elements and with

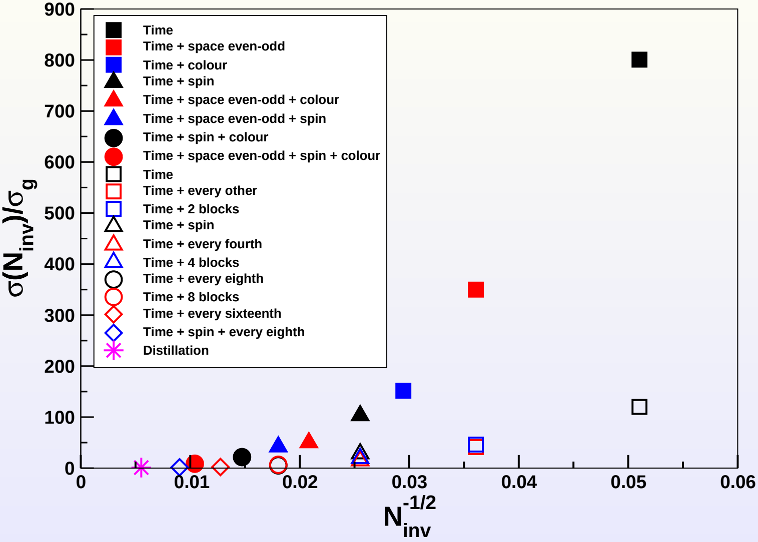
$$E[\eta_i] = 0 \text{ and } E[\eta_i \eta_j^*] = \delta_{ij}$$

- Now the distillation operator is written

$$\square = E[V\eta\eta^\dagger V^\dagger] = E[WW^\dagger]$$

- Introduces noise into computations
- **Dilution:** “thin out” the stochastic noise with N_η orthogonal projectors to make a variance-reduced estimator of $I_{\mathcal{D}} = E[WW^\dagger] = \sum_{k=1}^{N_\eta} E[V\mathcal{P}_k\eta\eta^\dagger\mathcal{P}_k V^\dagger]$, with $W_k = V\mathcal{P}_k\eta$, a $N_\eta \times (N_s \times N_c)$ matrix

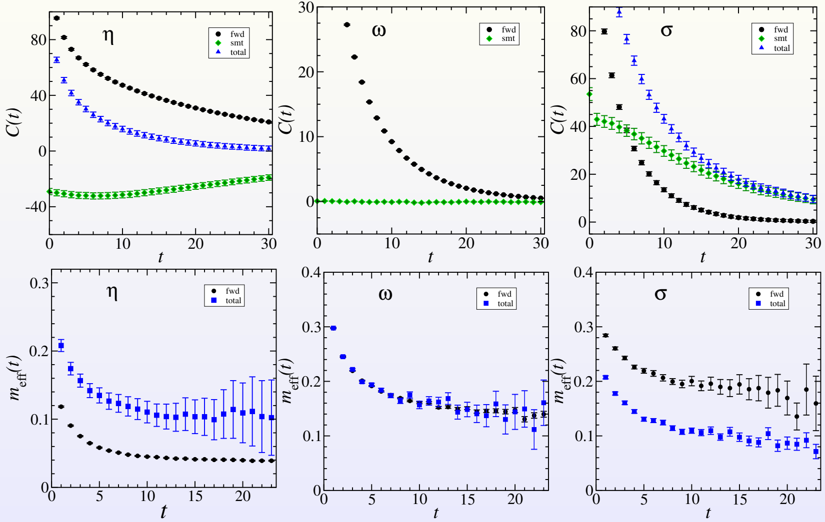
Stochastic estimation: baryon correlator



- Convergence faster for noise in distillation space

[arXiv:1011.0481]

Stochastic estimation: $I = 1, 0$ mesons



- propagation from $t - t$ is estimated differently from $t - t'$

[arXiv:1101.5398v1]

Summary

- Quarks are **difficult to manipulate directly** on the computer, but can be integrated analytically
  - The fields in the action → $\det M$
  - The fields in the operator → M^{-1}
- Need good algorithms and new ideas for **both** effects
- Quark propagator is too large to store: either **compute one column of it, or estimate it stochastically**
- Don't need all entries to make hadrons - redefine smearing
- Distillation is a promising method for making hadrons
- Works well, but it is expensive and scales poorly
- Stochastic estimators rescue us again