QCD at Finite Chemical Potential on the Lattice

2nd Day

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Asian School

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1.Formulation
2.Sign Problem

1st day

Trials to escape
from Sign Problem

2nd day

Some Topics

3rd day

Canonical Approach

In the school,
I heard the grand-
canonical and ??

Canonical Approach

- Miller and Redlich
  - Phys. Rev. D35 (1987) 2524
- A.Hasenfratz and Toussaint
  - Nucl.Phys.B371 (1992) 539
- Engels, Kaczmarek, Karsch and Laermann
  - Nucl.Phys. B558 (1999) 307 (hep-lat/9903030)
  - hep-lat/9905022
- Forcrand and Kratochvila
  - Nucl. Phys. B (P.S.) 153 (2006) 62 (hep-lat/0602024)
- A.Li, Meng, Alexandru, K-F. Liu
  - PoS LAT2008:032 and 178 (arXiv:0810.2349, arXiv: 0811.2112)

Grand-Canonical Z_{GC} ⇔ Canonical Z_C

$$\int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\beta S_G - \bar{\psi}\Delta\psi}$$

$$\hat{N} \equiv \int d^3x \bar{\psi}(x)\gamma_0\psi(x)$$

$$\delta(\hat{N} - Q) = \int d\phi\, e^{i\phi(\hat{N}-Q)}$$

We set $\phi = \dfrac{\mu_I}{T}$, the RHS is

$$\frac{3}{2\pi}\int_{-\frac{\pi}{3}}^{+\frac{\pi}{3}} d\frac{\mu_I}{T}\, e^{-iQ\frac{\mu_I}{T}}\, Z_{GC}(i\mu_I)$$

The integration region comes from the periodicity $\dfrac{2\pi T}{3}$ in μ_I

(Roberge-Weiss)

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Grand-Canonical Z_{GC} ⇔ Canonical Z_C

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(Roberge-Weiss)

Forcrand- Kratochvila
Nucl. Phys. B (P.S.) 153 (2006) 62
hep-lat/0602024

$m_\pi = 300\mathrm{MeV}$ KS Fermions

$6^3 \times 4$ $N_f = 4$

Kentucky's Trick

$$Z_C(Q) = \int \frac{d\phi}{2\pi} e^{-iQ\phi} Z_{GC}(\mu = i\pi T)$$

$$= \int \mathcal{D}U e^{-\beta S_G} \boxed{\int \frac{d\phi}{2\pi} e^{-iQ\phi} \det \Delta(\mu = i\phi T)}$$

How to calculte here？

$$\det \Delta = e^{\mathrm{Tr} \log \Delta}$$

We expand log

$$\mathrm{Tr} \log \Delta = A_0 + \sum_n A_n \cos(n\phi + \delta_n)$$

n: winding number

can be expressed by Bessel functions.

A.Li, Meng, Alexandru, K-F. Liu
PoS LAT2008:178 (arXiv:0810.2349)

$$N_f = 4$$

$$6^3 \times 4$$

$$m_\pi = 1\mathrm{GeV}$$

Wilson Fermions

Kentucky 2010

李安意， Alexandru, 刘克非， 孟祥飞

$$6^3 \times 4$$

$$m_\pi = 700 \sim 800 \qquad N_f = 4$$

For $N_f = 2$

$T = 0.83T_c,\ 0.87T_c,\ 0.92T_c$

No clear signal for S-shape

Density of State

- Gocksch
  - Phys. Rev. Lett. 61 (1988) 2054.
- Anagnostopoulos and Nishimura
  - Phys.Rev. D66 (2002) 106008 (hep-th/0108041)
- Fodor, Katz and Schmidt
  - JHEP0703:121,2007 (hep-lat/0701022)

So far the largest density is investigated by this method.

DOS Simplest Case

$$\int \mathcal{D}U e^{-S(U)}$$

$\delta(E - S(U))$

$$\rho(E) = \int \mathcal{D}U \delta(E - S(U)) e^{-S(U)}$$

(Density of State)

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) \langle O \rangle_E$$

$$\langle O \rangle_E \equiv \frac{1}{\rho(E)} \int \mathcal{D}U \delta(E - S) e^{-S} O(U)$$

$$Z_\rho \equiv \int dE \rho(E)$$

DOS General Form.

Muroya, Nonaka, Takaishi, AN, PTP,10, (2003) 615
(hep-lat/0306031) Sec.5.5

$$\rho(E) = \int dU \delta(E - x(U)) g(U)$$

any ! (∗)

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) \langle O \det \Delta e^{-\beta S_g} / g(U) \rangle_E$$

$$Z_\rho = \int dE \rho(E) \langle \det \Delta e^{-\beta S_g} / g(U) \rangle_E$$

$$\langle O \rangle_E = \frac{1}{\rho(E)} \int dU \delta(E - x(U)) g(U) O(U)$$

(∗) As g(U), we may use some approximte form of the fermion determinant.

DOS: The phase is taken as a variable. (Gocksch)

$$x(U) = \theta(U)$$ Phase of $\det \Delta$

$$g(U) = |\det \Delta(\mu)| e^{-\beta S_g}$$

$$\rho(E) = \int dU \delta(E - \theta(U)) |\det \Delta(\mu)| e^{-\beta S_g}$$

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) e^{iE} \langle O \rangle_E$$

DOS: Plaq.is taken as a variable.

Strong Coupling Calculation

$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{-\bar{\psi}\Delta\psi - \cancel{\frac{6}{g^2} S_G}}$$

Then we can integrate over U.
(Time-like U and Spatial U are separated.)

Bilic et al. Nucl. Phys. B377 (92)615

$$Z = \int dU d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 e^{\bar{\psi}_1 U \psi_2 - \bar{\psi}_2 U^\dagger \psi_1}$$

$$Z = 1 + \frac{1}{3} M_1 M_2 + (-\bar{B}_1 B_2 + \bar{B}_2 B_1) + \cdots$$

$$\sim e^{\frac{1}{3} M_1 M_2 + (-\bar{B}_1 B_2 + \bar{B}_2 B_1)}$$

$$M_i \equiv \bar{\psi}_i^a \psi_i{}^a$$

$$B_i \equiv \frac{1}{3!} \epsilon_{a_1 a_2 a_3} \psi_i{}^{a_1} \psi_i{}^{a_2} \psi_i{}^{a_3}$$

Many useful formulae in
Rossi and Wolff, Nucl.Phys. B248 (1984) 541

Strong Coupling Calculation (cont)

- Revival:
  - Nishida, Fukushima and Hatsuda, Phys.Rept. 398 (2004) 281 (SU(2))
  - Nishida, PRD69(04)094501(SU(3)), hep-ph/0310160(SU(2))
    - KS-fermions, including the di-quark condensation
    - Finite-Isospin is also considered (8-flavors)

SU(2)

Ohnishi (Hokkaido-->Kyoto) Group

- Since 2005, Strong coupling expansion
  - SU(3), KS fermions, until next-next-leading order

Until Next-to-Next-to-Leading Order

Quarkynic phase is observed ?

Fromm and deForcrand

Phys.Rev.Lett.104:112005,2010

- Aloisio et al, Nucl.Phys. B564, 489(2000)
  - the ergodicity of the previous monomer flip simulations are questionable.
- “snake” algorithm

Complex Langevin

- Parisi
  - Phys. Lett. 131B (1983),393
- Klauder
  - Phys. Rev. A29 (1984),2036
- Klauder and Petersen
  - J. Stat.Phys., 39 (1985),53
- Karsch and Wyld
  - Phys. Rev. Lett. 21(1985),2242
- Ambjørn and Yang
  - Phys. Lett. B165, (1985)140
- Matsui and A.N
  - Phys. Lett. B194 (1987),262
- Aarts and Stamatescu
  - JHEP 0809:018,2008 (arXiv:0807.1597)

Basic Idea

Canonical Quantication ⟷ Path Int. ⟷ Stochastic Quantization

$$P \propto e^{-S}$$

$$\frac{dx(\tau)}{d\tau} = -\frac{dS}{dx} + \eta(\tau)$$

η : Quantum Noise

τ :Stochastic time

No Probability！

We can solve this equation even for a complex S
(But there is a corresponding Fokker-Planck Eq.)

Problem：Run-Away Solutions,

Wrong Convergence

Aarts and Stamatescu

JHEP 0809:018,2008 (arXiv:0807.1597)

Successful !!
Why ?
Adaptive step size ?
Really ?
Very simple model. (Hopping expansion)

QCD Inequality

What is the relation ?

$$\mathrm{Tr} AB^\dagger \leq \sqrt{\mathrm{Tr} AA^\dagger}\sqrt{\mathrm{Tr} BB^\dagger}$$

(Cauchy-Schwarz)

Meson $\bar{\psi}\Gamma\psi$

Propagators

$$\mathrm{Tr}\Delta^{-1}(x,0)\Gamma\Delta^{-1}(0,x)\Gamma$$

$$= \mathrm{Tr}\Delta^{-1}(x,0)\Gamma\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5\Gamma$$

$$\leq \sqrt{\Delta^{-1}(x,0)\Delta^{-1}(x,0)^\dagger}$$

$$\sqrt{\Gamma\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5\Gamma\left(\Gamma\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5\Gamma\right)^\dagger}$$

$$= \mathrm{Tr}\Delta^{-1}(x,0)\Delta^{-1}(x,0)^\dagger$$

π Propagators

(Any Meson Propagators) $\leq$ (pi Meson Propagators)

$m_\pi \leq m$

π is lightest.

This does not hold for $\mu \neq 0$

Color SU(2)

- No Sign Problem → MC simulation is possible.
- There are Confinement/Deconfinement Phases.
- No difference between Meson and Baryon.
- A platform to study finite density field theory
- A Laboratory to study fermion effects based on field theory.

Old(est) Reference

- A.Nakamura
  Behavior of quarks and gluons at finite temperature and density in SU(2) QCD
  Phys.Letters 149B, pp391--395, 1984

$\kappa = 0.160$

Muroya, A.N. and Nonaka
Phys.Lett. B551 (2003) 305
(hep-lat/0211010)

di-quark source

Hands, Sitch and Skullerud
Phys.Lett. B662, (2008)405
(arXiv:0710.1966)

Using Effective Theories ?

- Lattice EFT ?

Lattice EFT for nucleons

Finite volume matching for two-nucleon states
For the same periodic volume, compute two-nucleon energies
in Lattice QCD and match to two-nucleon energies Lattice EFT

QCD critical point in the PNJL model as a function of the uncontrolled model parameters

Fukushima QM08

Vector-channel interaction dependence

P-NJL vs.Lattice (Kyushu group)

Hadron Masses in P-NJL

Starting from an Approximation ?

- Condenst Matter Physics, a typical method is to calculate deviation numerically from the mean field.
- `Mean Field' in our case ?
    - Hopping parameter ?
    - Strong coupling ?

Backup Slides

Taylor Expansion of Screening Mass *M*
QCD-TARO Collaboration

Phys. Rev. D65 (2002) 054501
Phys. Lett. B609 (2005) 265-270

$$\frac{M(\mu)}{T} = \left.\frac{M}{T}\right|_{\mu=0} + \left.\left(\frac{\mu}{T}\right)\frac{\partial M}{\partial \mu}\right|_{\mu=0} + \frac{1}{2}\left(\frac{\mu}{T}\right)^2 \frac{T}{2}\left.\frac{\partial^2 M}{\partial \mu^2}\right|_{\mu=0} + O\left[\left(\frac{\mu}{T}\right)^3\right]$$

$$\left.\frac{\partial M}{\partial \mu}\right|_{\mu=0}, \left.\frac{\partial^2 M}{\partial \mu^2}\right|_{\mu=0}$$

M : **Screening mass**

$$M^2_{screen} = m^2 + \Pi(0, \vec{p}, T)$$

$$M^2_{pole} = m^2 + \Pi(p_0, \vec{0}, T)$$