QCD at Finite Chemical Potential on the Lattice 3rd Day

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Asian School
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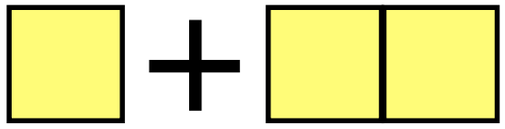
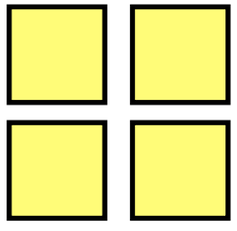
1. Reduction Formula
2kuhhjkhjk. Imainary Chemical Potential

Status Report of Our Project

- Lattice QCD with Wilson Fermions
  - Improved actions
    - Gluonic part: 
    - Quark part: 
  - Real and Imaginary Chemical Potentials

Studies of Finite Density QCD (SU(3)) with Wilson Fermions

- H.-S. Chen, X.-Q. Luo
  - Phys.Rev. D72 (2005) 0345041
  - hep-lat/0411023
- A.Li, X. Meng, A. Alexandru, K-F. Liu
  - PoS LAT2008:032 and 178 (arXiv:0810.2349, arXiv: 0811.2112)
- C. Gattringer and L. Liptak
  - arXiv:0906.1088
- J. Danzer, C. Gattringer, L. Liptak and M. Marinkovic
  - arXiv:0907.3084 and LAT2009: 185 (2009) (arXiv: 0910.3541)

In the finite density lattice QCD,

- we should often handle the fermion determinant, directly,
  - e.g.
    - Multi-parameter Re-weighting by Fodor-Katz

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det \Delta(\mu) \, e^{-\beta S_G}$$

$$= \frac{1}{Z} \int \mathcal{D}U \, O \, \underbrace{\det \Delta(0) \, e^{-\beta_0 S_G}}_{\text{Measure}} \, \underbrace{\frac{\det \Delta(\mu)}{\det \Delta(0)} \, e^{(\beta_0 - \beta) S_G}}_{\text{Reweighting Factor}}$$

For KS Fermions, a Trick behind

- Gibbs Formula(*)
  - P.E.Gibbs, Phys.Lett. B172 (1986) 53-61

$$
\begin{aligned}
\det \Delta &= z^{-N} \begin{vmatrix} -B(-V) - z & 1 \\ -V^2 & -z \end{vmatrix} \\
&= \left| \boxed{\begin{pmatrix} BV & 1 \\ -V^2 & 0 \end{pmatrix}} - zI \right| \\
&= \det\left(P - zI\right) \\
&= \prod \left(\lambda_i - z\right)
\end{aligned}
$$

P

$$z \equiv e^{-\mu}$$

- P is $(2 \times N_c \times N_x \times N_y \times N_z)^2$ (Matrix Reduction)
- Determinant for any value of μ

*) A similar formula was developped by Neuberger (1997) for a chiral fermion and applied by Kikukawa(1998).

A Reduction Formula for Wilson Fermions

- Keitaro Nagata and Atsushi Nakamura
  Wilson Fermion Determinant in Lattice QCD
  Phys. Rev. D82,094027 (arXiv:1009.2149)
- A. Alexandru and U. Wenger
  arXiv:1009.2197
- Budapest-Wuppertal group also obtained a similar result.

The same matrix transformation like KS case cannot be employed, due to the fact that

$r \pm \gamma_4$ have no inverse, if the Wilson term $r = 1$.

Gibbs started to multiply V to the fermion matrix Δ.

Instead, we multiply $P = (c_a r_- + c_b r_+ V z^{-1})$

Here,

$$V = \left(\begin{array}{c|c|ccc|c}
0 & U_4(t=1) & 0 & \cdots & & 0 \\ \hline
0 & 0 & U_4(t=2) & \cdots & & 0 \\ \hline
0 & 0 & 0 & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & \cdots & U_4(t=N_t-2) & 0 \\ \hline
0 & 0 & & \cdots & 0 & U_4(t=N_t-1) \\ \hline
-U_4(t=N_t) & 0 & & \cdots & 0 & 0
\end{array}\right)$$

c_a and c_b are arbitary non-zero numbers.

$$\det P = (c_a c_b z^{-1})^{N/2}$$

if we take the following trick, Borici (2004)

$$r_+ r_- = \frac{r^2 - 1}{4} = \epsilon \to 0$$

where $r_\pm \equiv \dfrac{r \pm \gamma_4}{2}$

After very long calculation (See Nagata-Nakamura arXiv:1009.2149), we get

$$\det \Delta(\mu) = (c_a c_b)^{-N/2} z^{-N/2}$$

$$\times \left(\prod_{i=1}^{N_t} \det(\alpha_i) \right) \det \left(z^{N_t} + Q \right)$$

$$\frac{\det \Delta(\mu)}{\det \Delta(0)} = \frac{\det(\xi + Q)}{\det(1+Q)}$$

$$\xi \equiv e^{-\mu/T}$$
(fugacity)

Q is $(4N_cN_xN_yN_z) \times (4N_cN_xN_yN_z)$ matrix.

No N_t !

$\left((2N_cN_xN_yN_z) \times (2N_cN_xN_yN_z) \right.$ KS case $\left. \right)$

Diagonalize Q,

$$Q \to \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_{N_{red}} \end{pmatrix}$$

$$\det(\xi + Q) = \prod (\xi + \lambda_n)$$

λ_n does not depend on μ.

Once we calculate λ_n,
we can evaluate $\det \Delta(\mu)$ for any μ.

Symmetry in Eigen Values

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

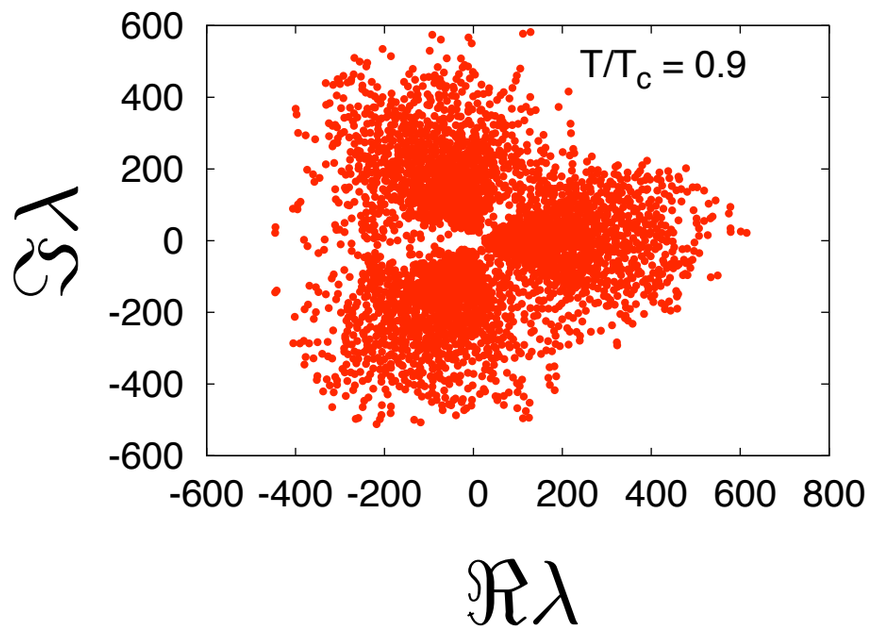
$$\det \Delta(\mu) \propto \prod (\xi + \lambda_n)$$

$$\xi \equiv e^{-\mu/T}$$

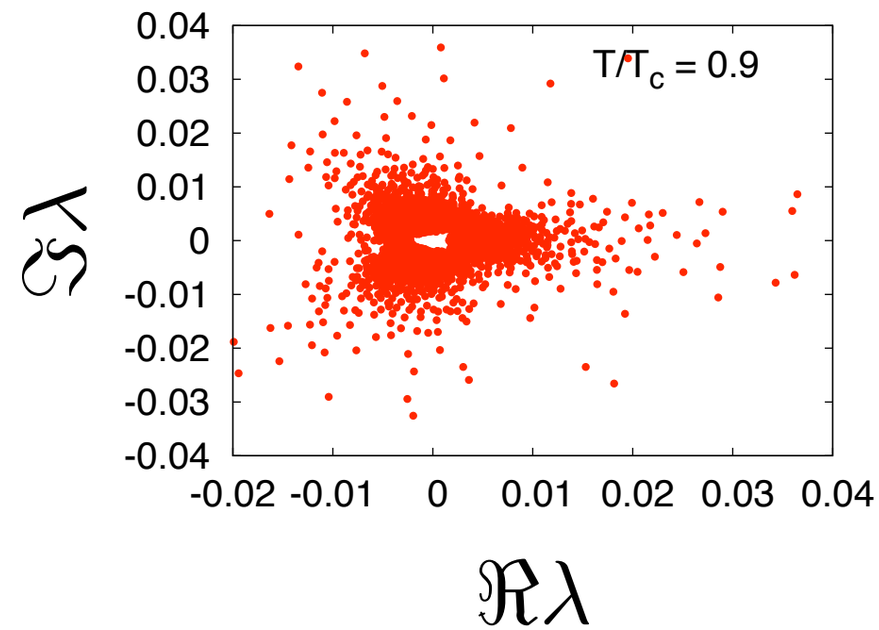
$$\lambda_{n'} = \frac{1}{\lambda_n^*}$$

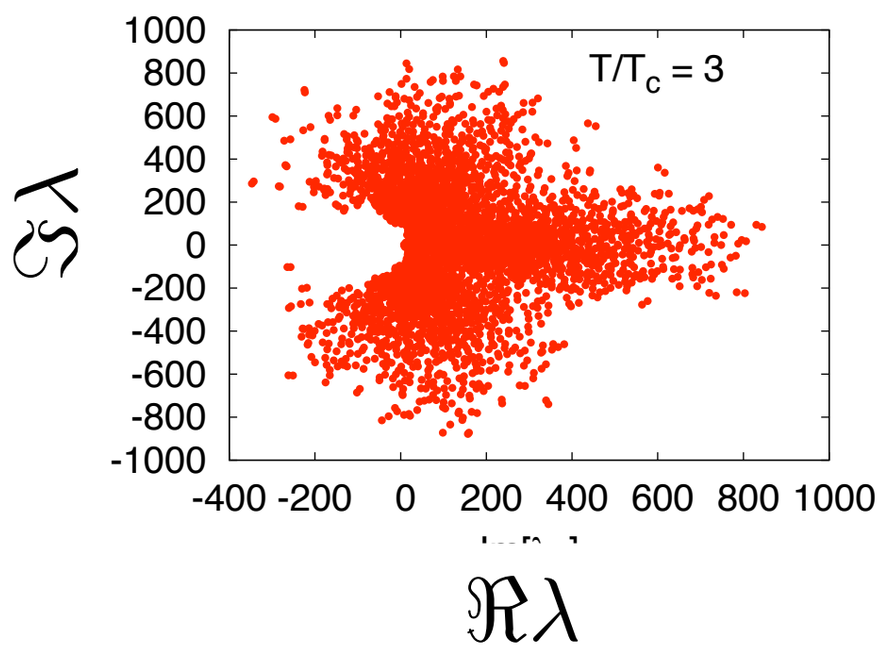
Eigen Value Distributions

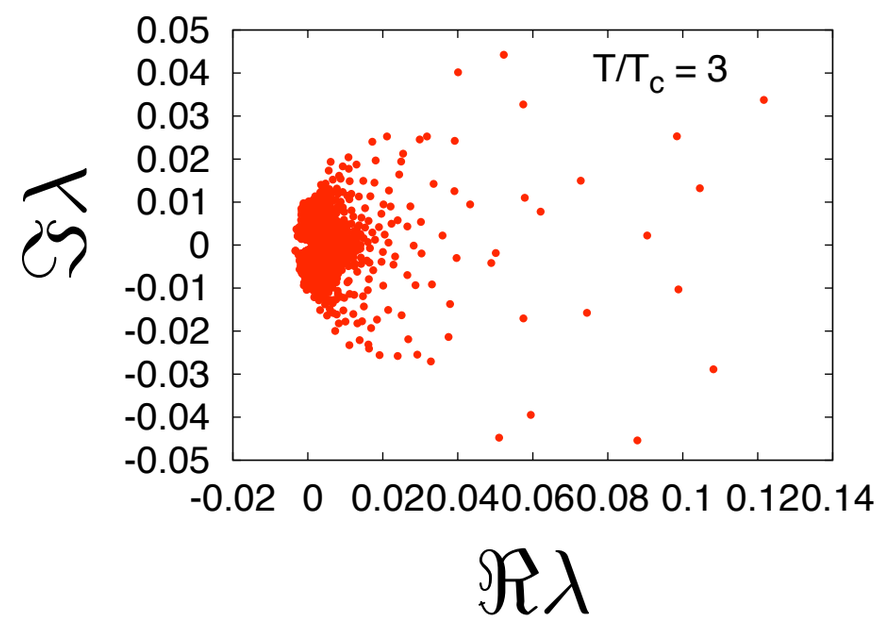
Large Eigenvalues

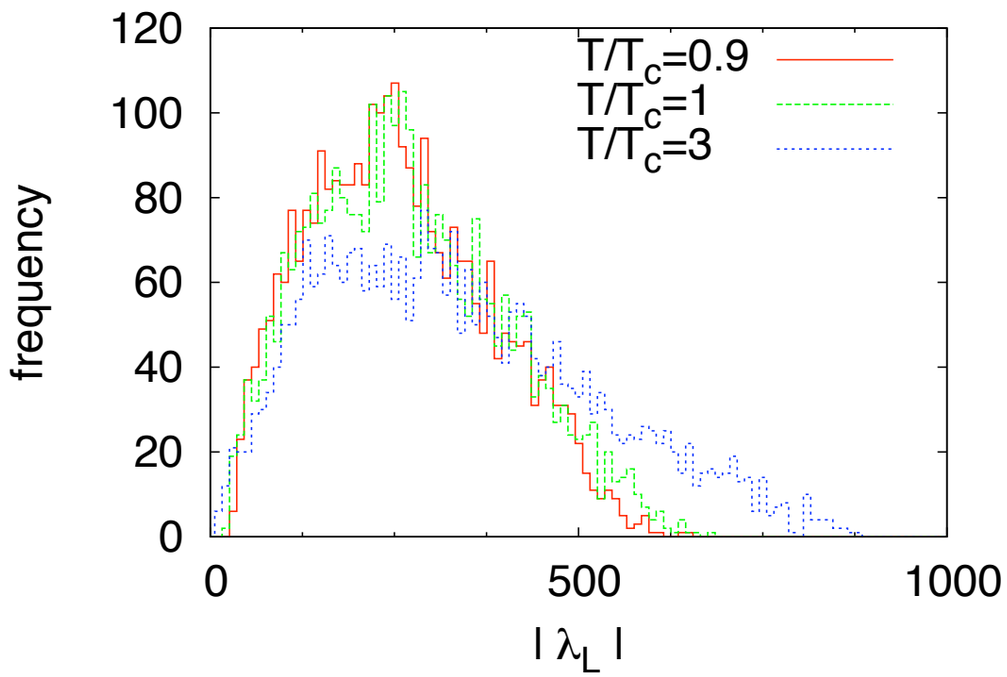


Small Eigenvalues





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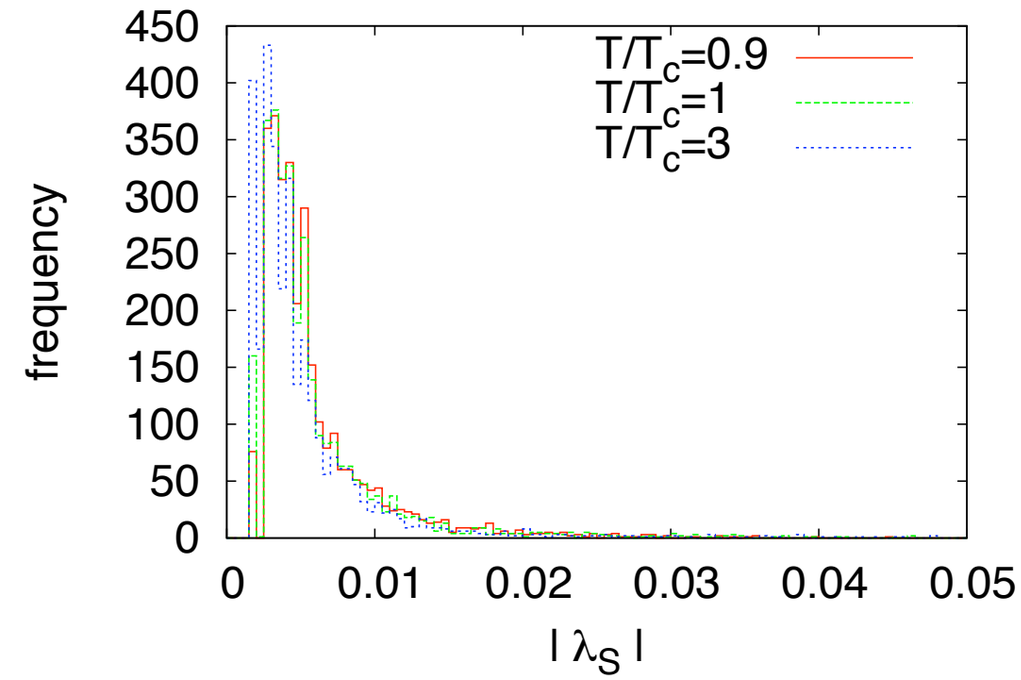
80
60
40
20
0
0
500
1000
| λL |
T/Tc=0.9
T/Tc=1
T/Tc=3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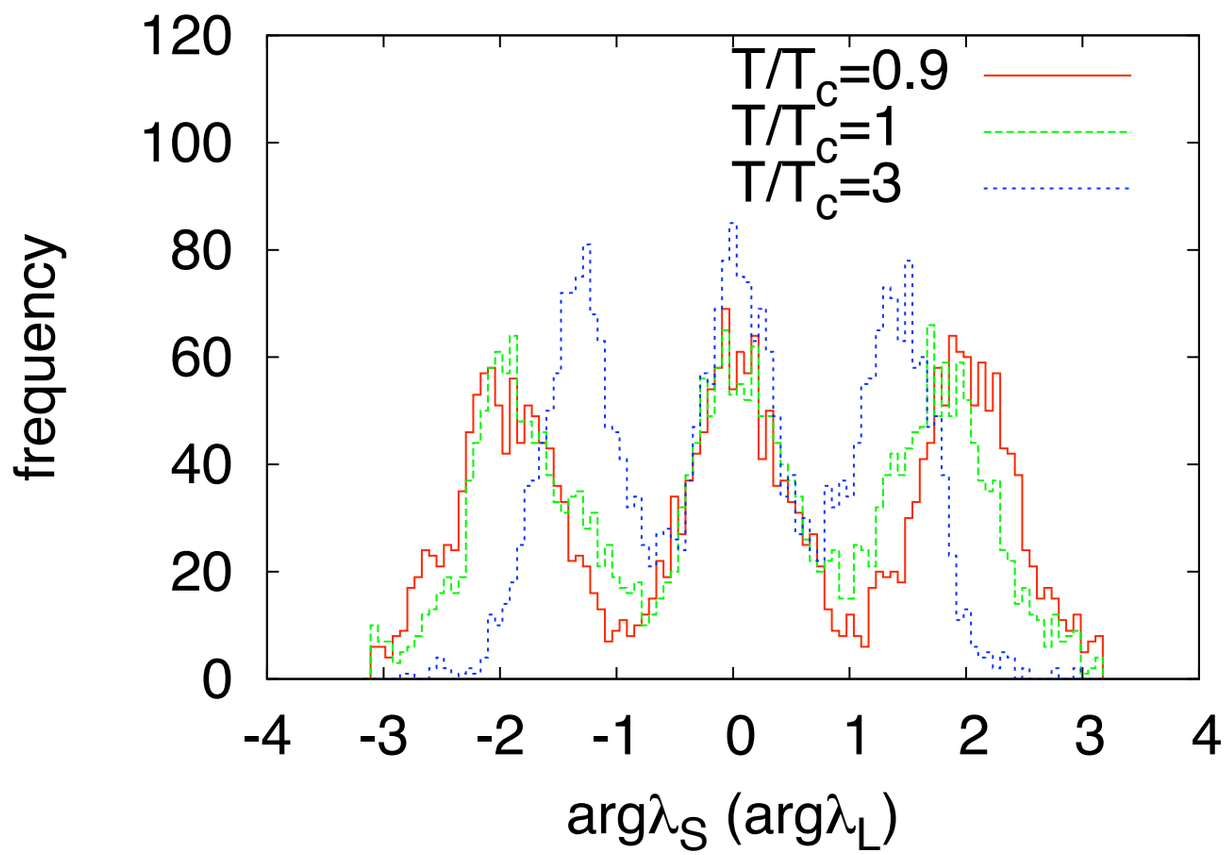
0.05
| λS |
T/Tc=0.9
T/Tc=1
T/Tc=3


frequency
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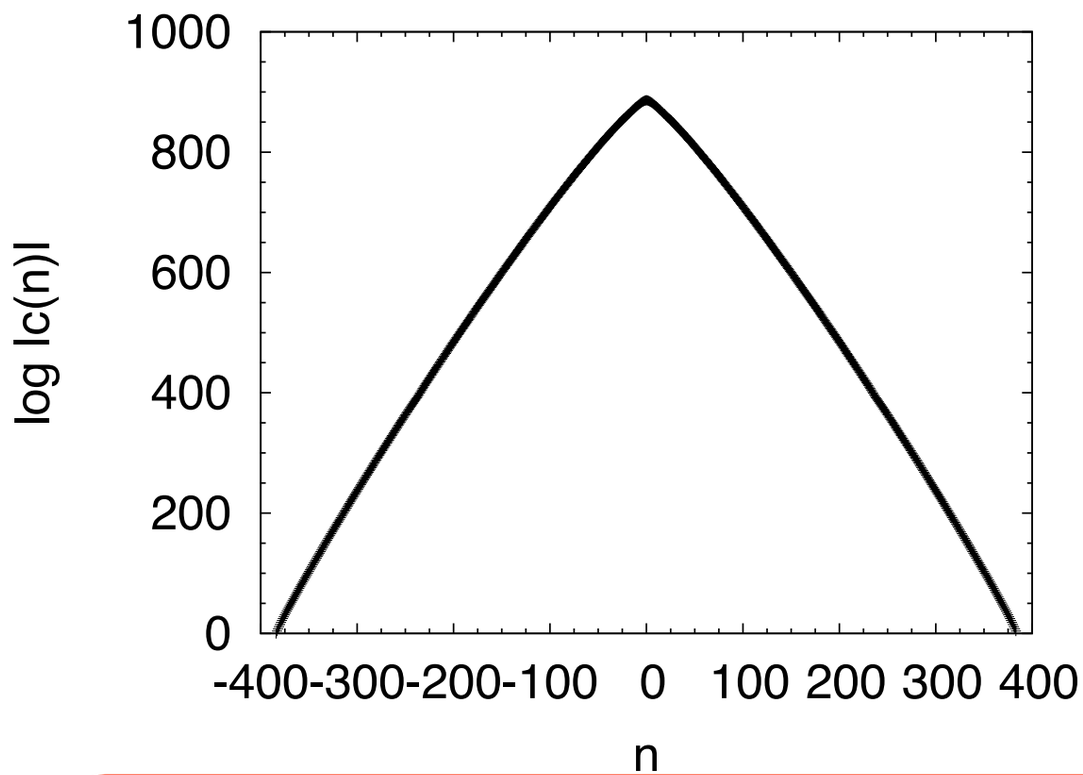
100
80
60
40
20
0
-4
-3
-2
-1
0
1
2
3
4
argλS (argλL)
T/Tc=0.9
T/Tc=1
T/Tc=3

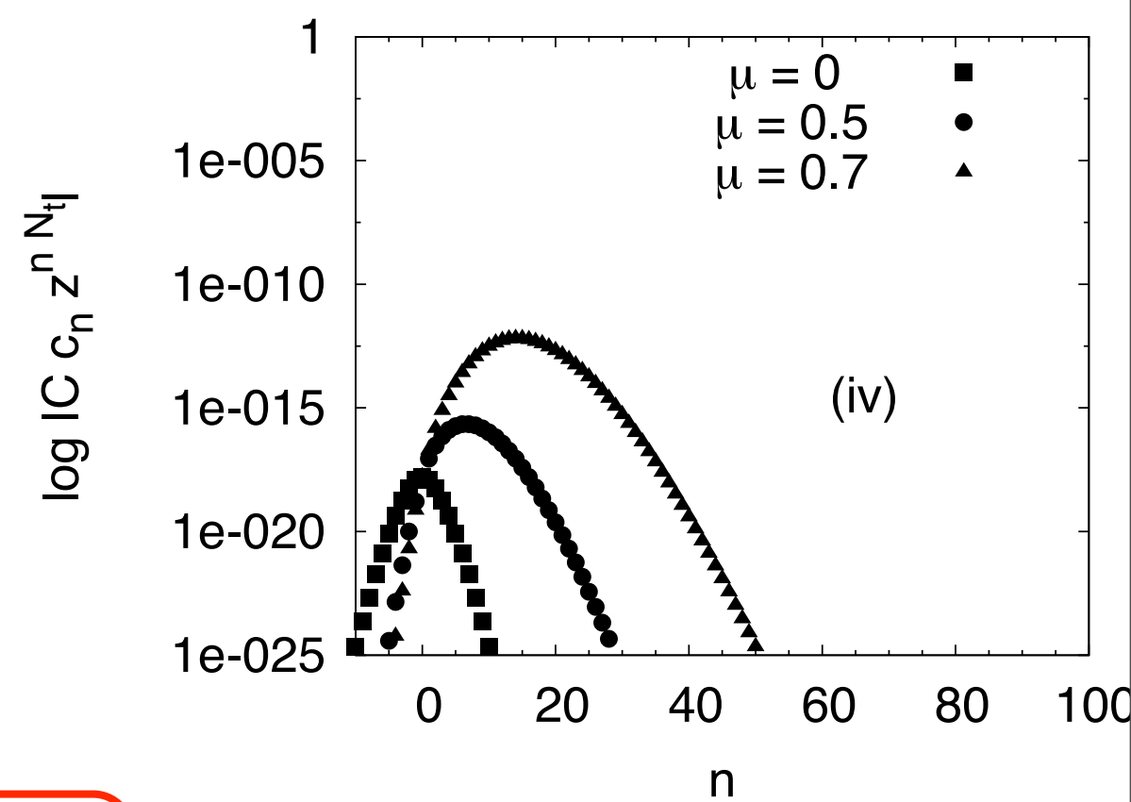
$$\det(\xi + Q) \;=\prod(\xi+\lambda_k) \;=\sum C_n \xi^n$$

$$\beta = 1.85 \quad 4^4$$

$$\log|C_n|$$



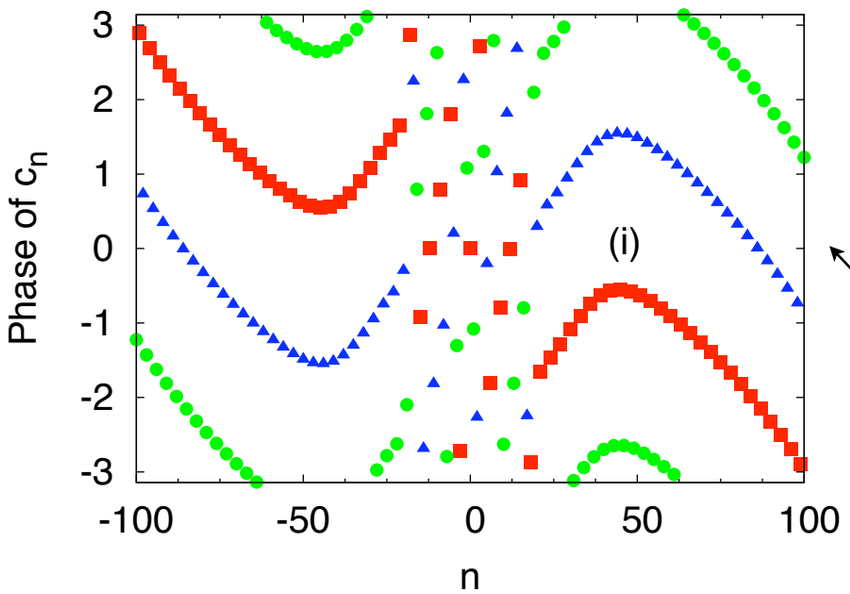
$$\log|C_n|\left(e^{\mu/T}\right)^n$$



To handle these very wide range numbers, we need special care on the accuracy.

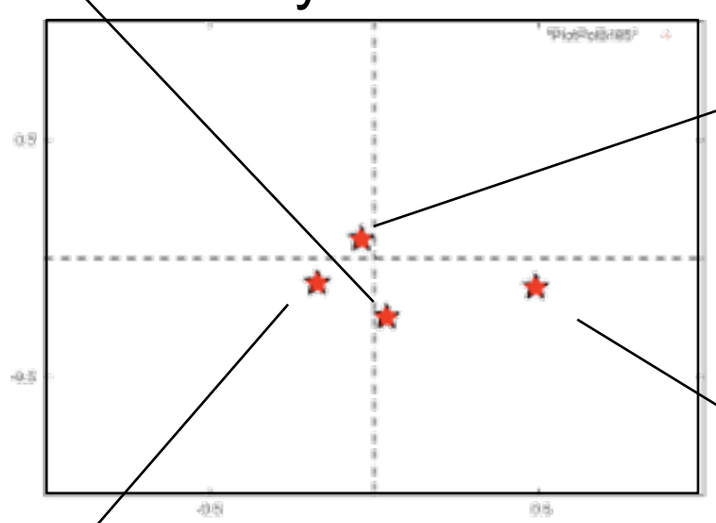
Phase of C_n

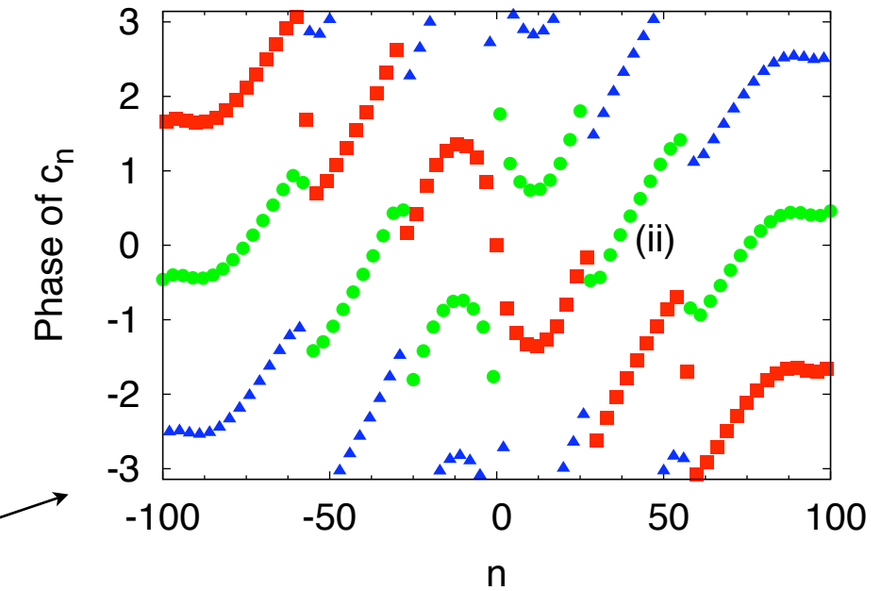
$$4^4 \quad \beta = 1.85 \quad (T \sim T_c)$$

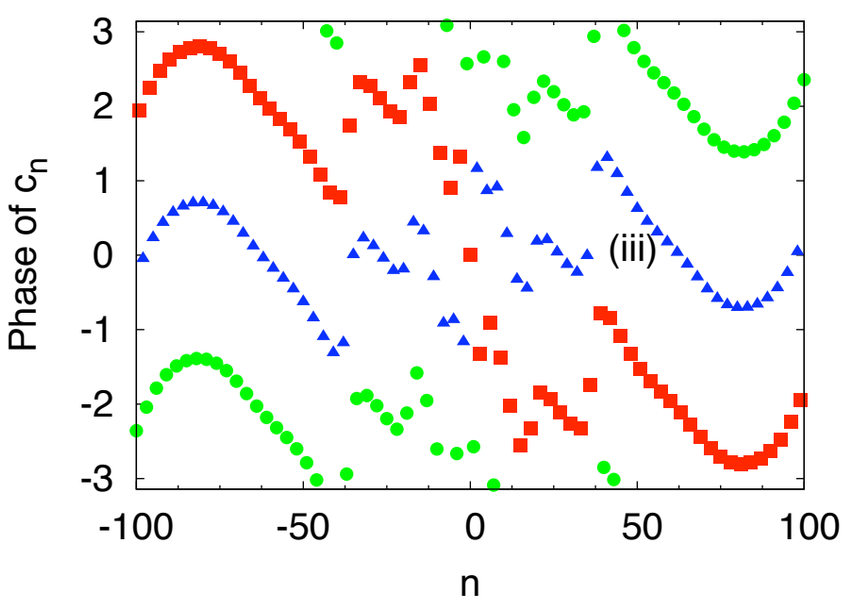


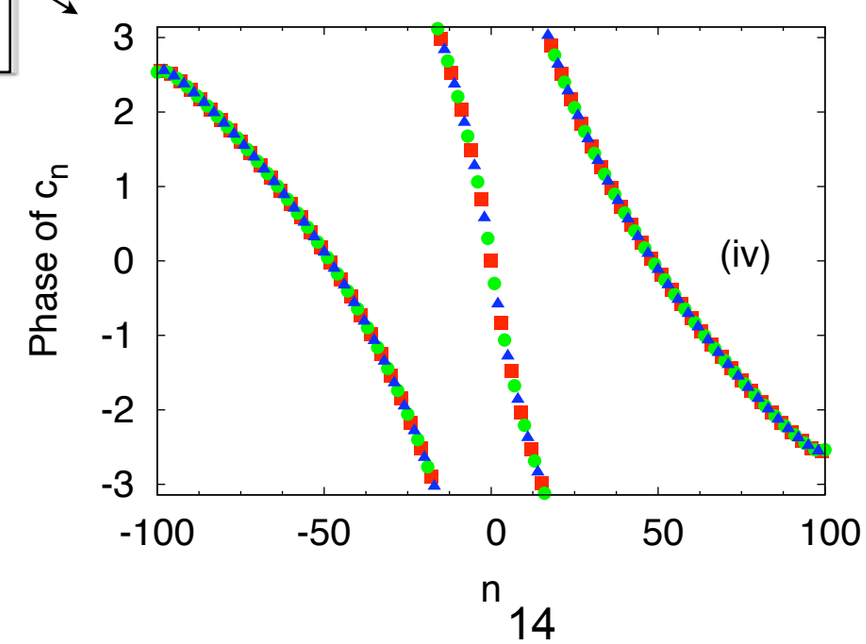
$n = 3m$
$n = 3m + 1$
$n = 3m + 2$

Polyakov Line





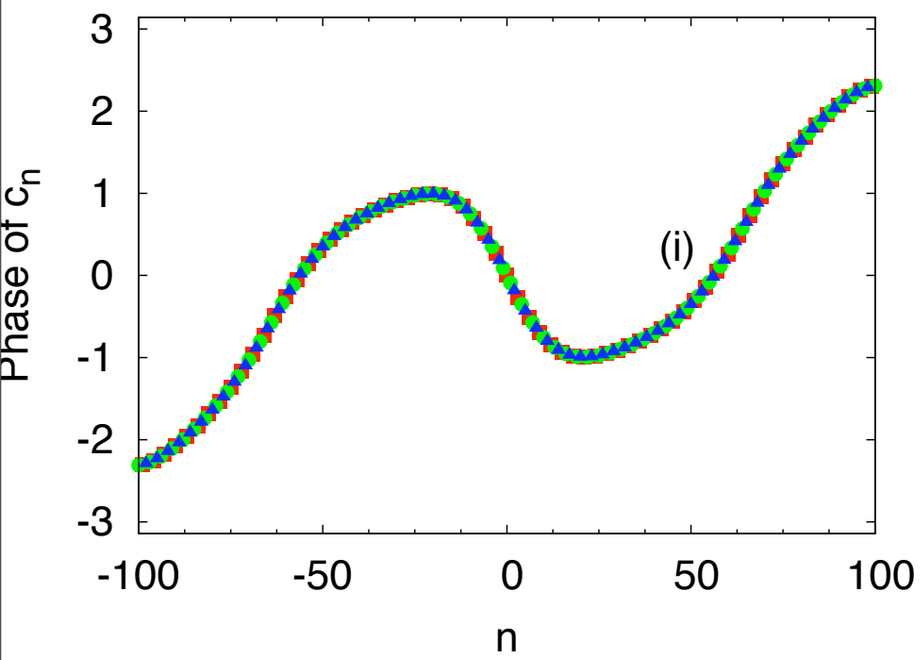


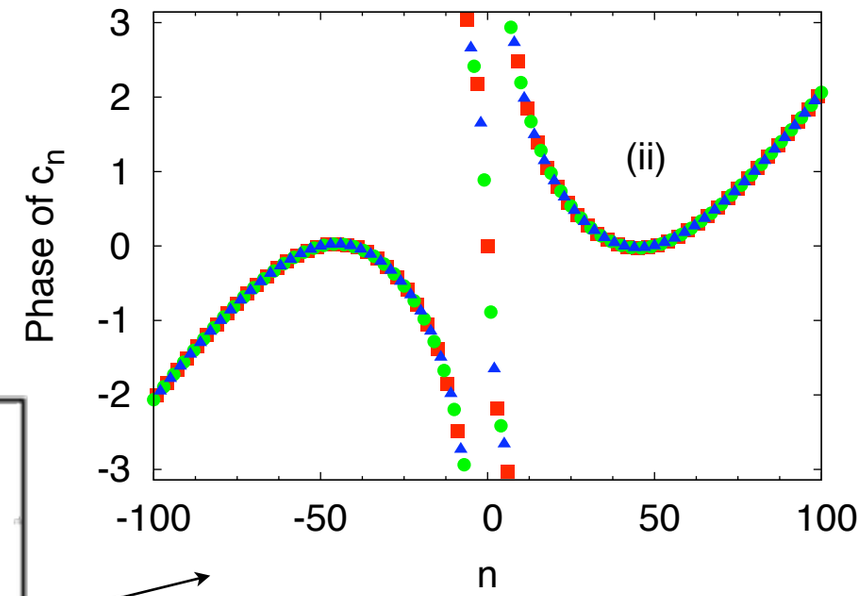


Phase of

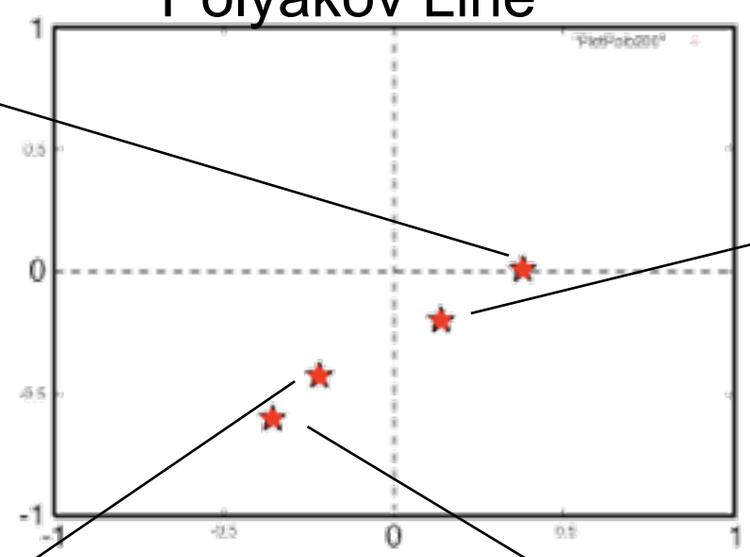
$$4^4 \quad \beta = 2.0 \qquad (T > T_c)$$

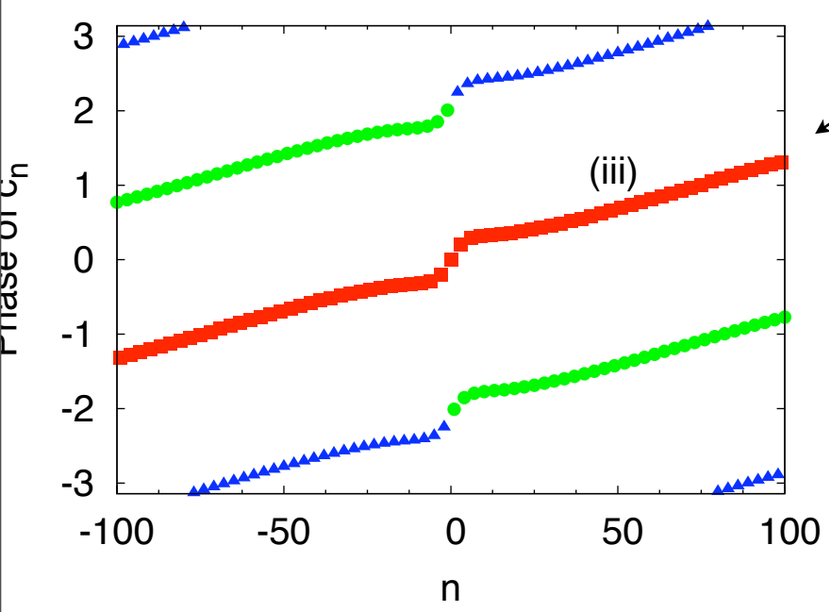
$n = 3m$
$n = 3m + 1$
$n = 3m + 2$

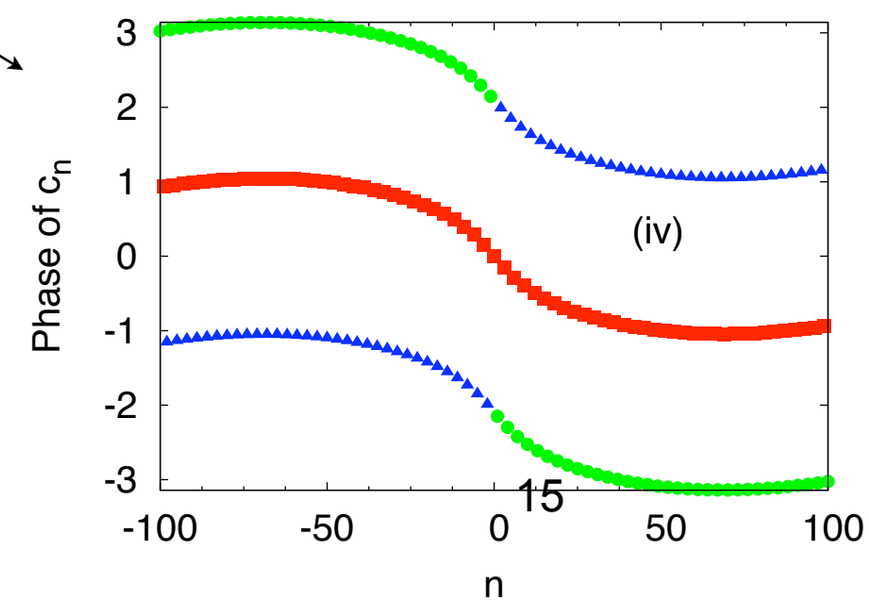




Polyakov Line







Reweighting Factor

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det \Delta(\mu) \, e^{-\beta S_G}$$

$$= \frac{1}{Z} \int DU O \underbrace{\det{}^2\Delta(0) e^{-\beta_0 S_G}}_{\text{Measure}} \underbrace{\mathcal{R}.\mathcal{F}.}_{\text{Reweighting Factor}} = \frac{< O \times R.F. >_0}{< R.F. >_0}$$

Here

$$\mathcal{R}.\mathcal{F}. = \left(\frac{\det \Delta(\mu)}{\det \Delta(0)} \right)^2 \times e^{(\beta_0 - \beta) S_G} \equiv e^{2\theta} e^{F} e^{G}$$

Reweighting Factor should be “LARGE”.

“LARGE” ? It is a function of *U*.

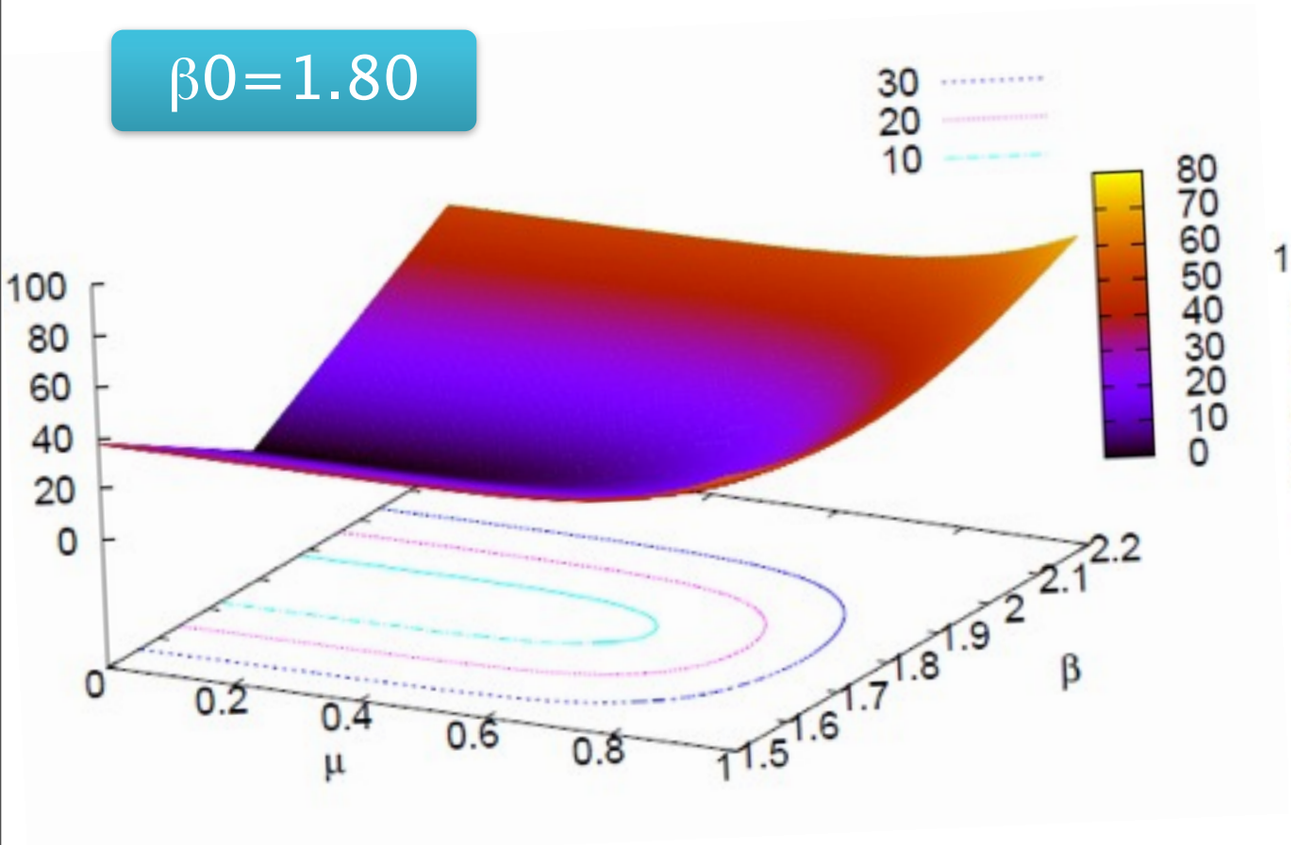
Large Contribution ?

Small Fluctuation ?

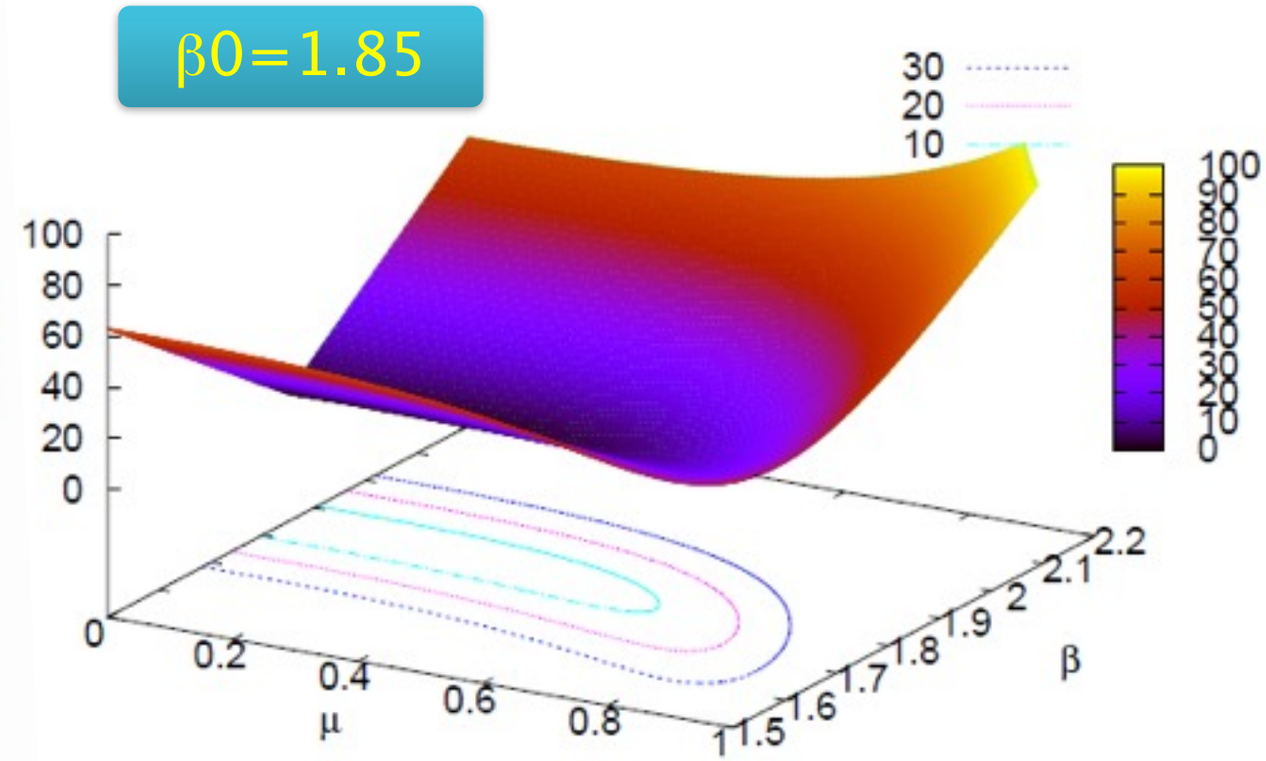
S.Ejiri, Phys.Rev. D69 (2004) 094506 hep-lat/0401012

$((F+G)-\langle F+G\rangle)^2$ $8^3\times 4$ *Preliminary*

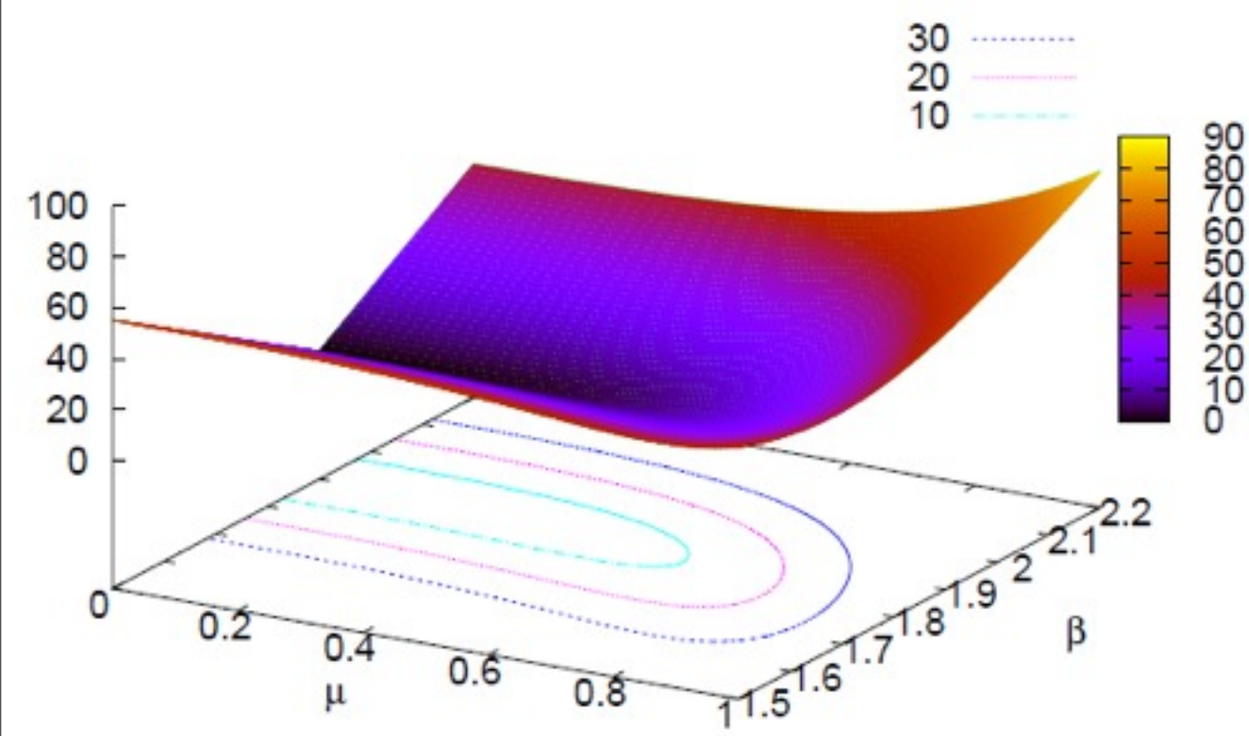
β0=1.80



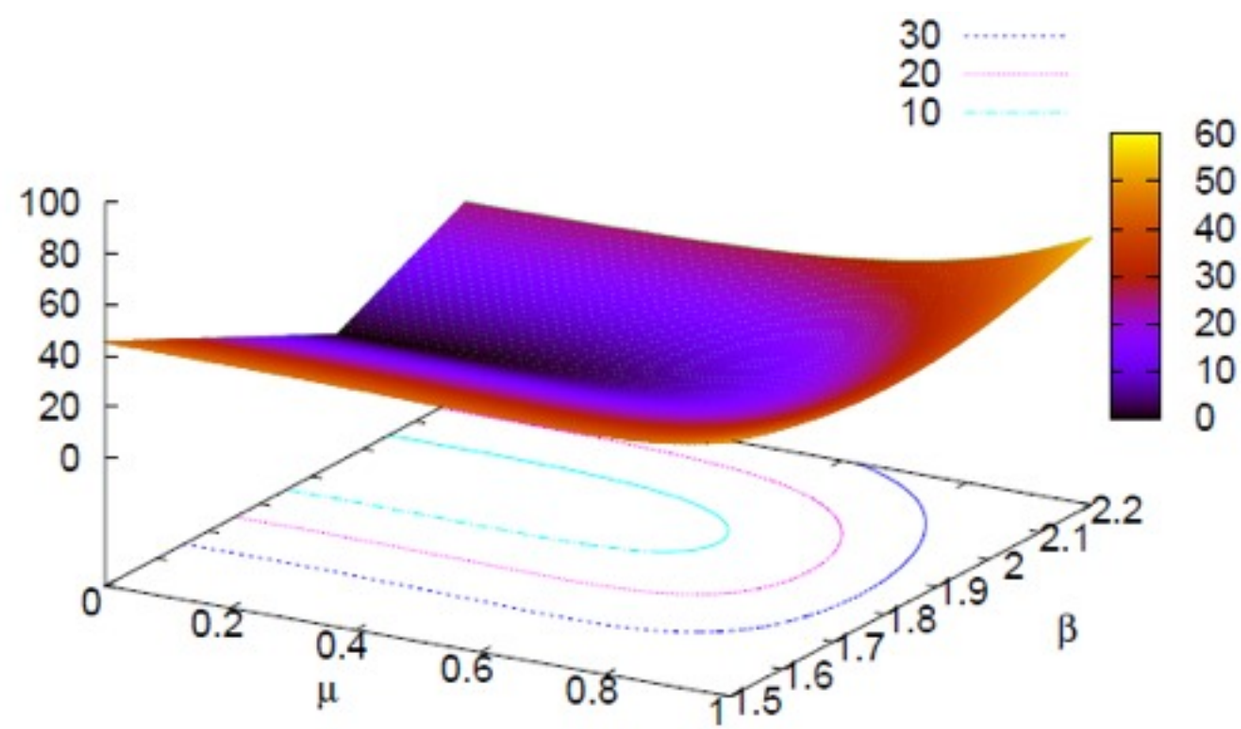
β0=1.85



β0=1.90



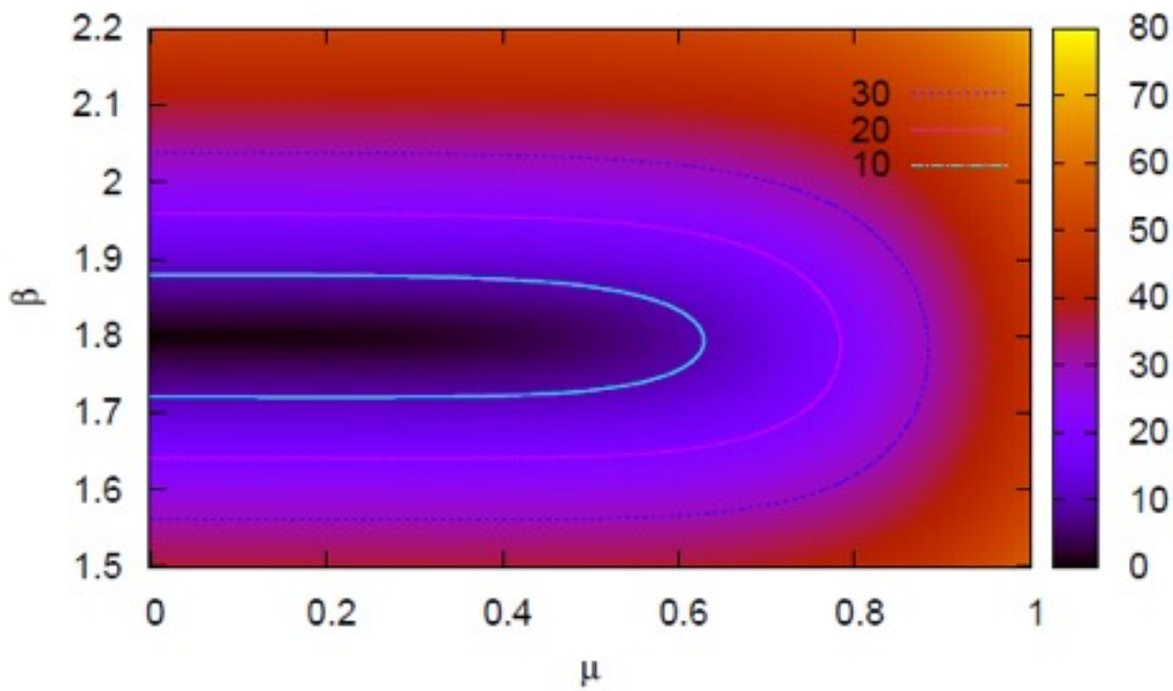
β0=1.95



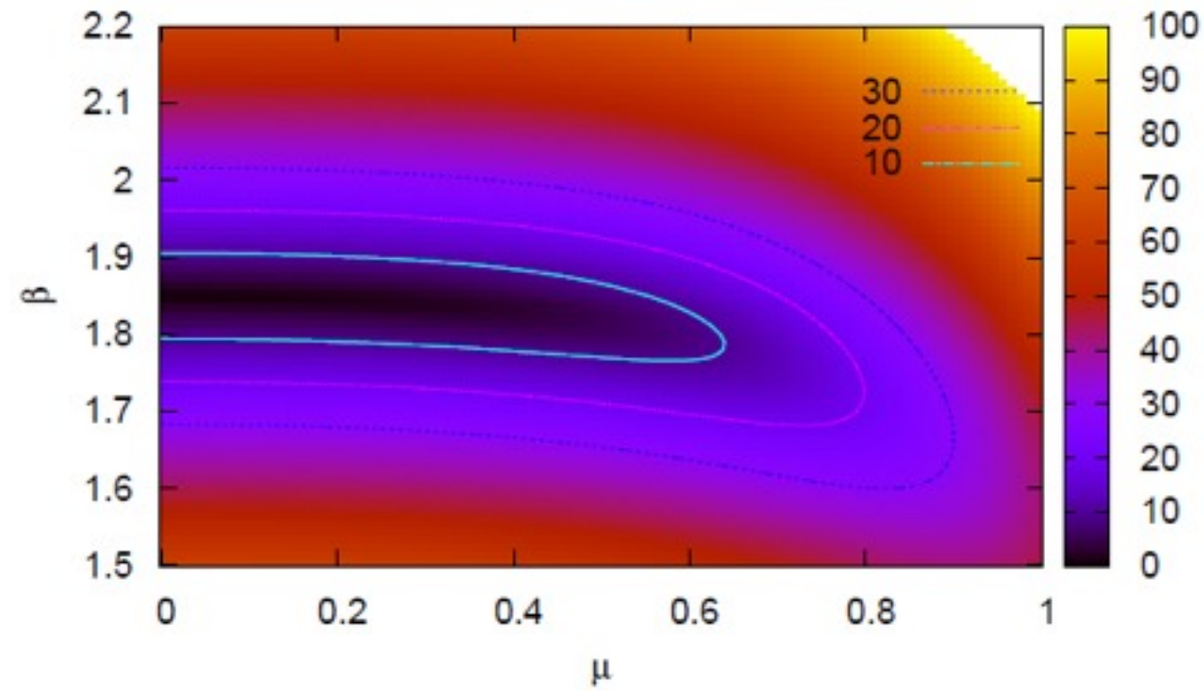
$$\langle\ (\ (F+G)-\langle F+G\rangle\ )^2\rangle \quad 8^3\times 4$$

Preliminary

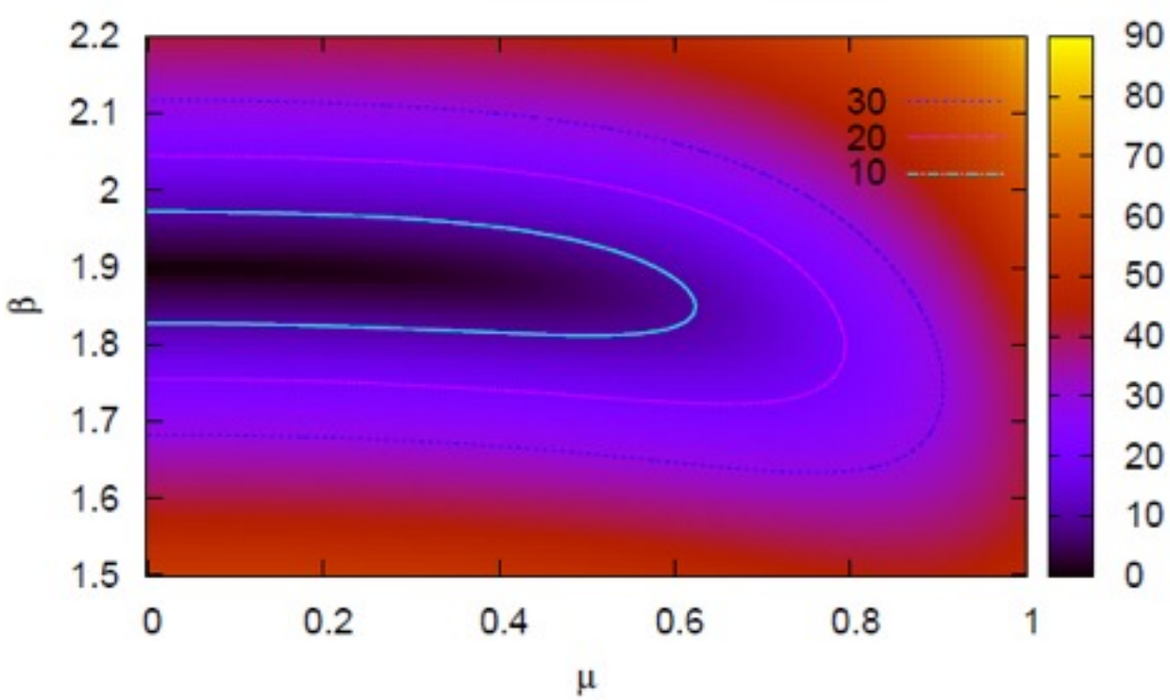
β0=1.80



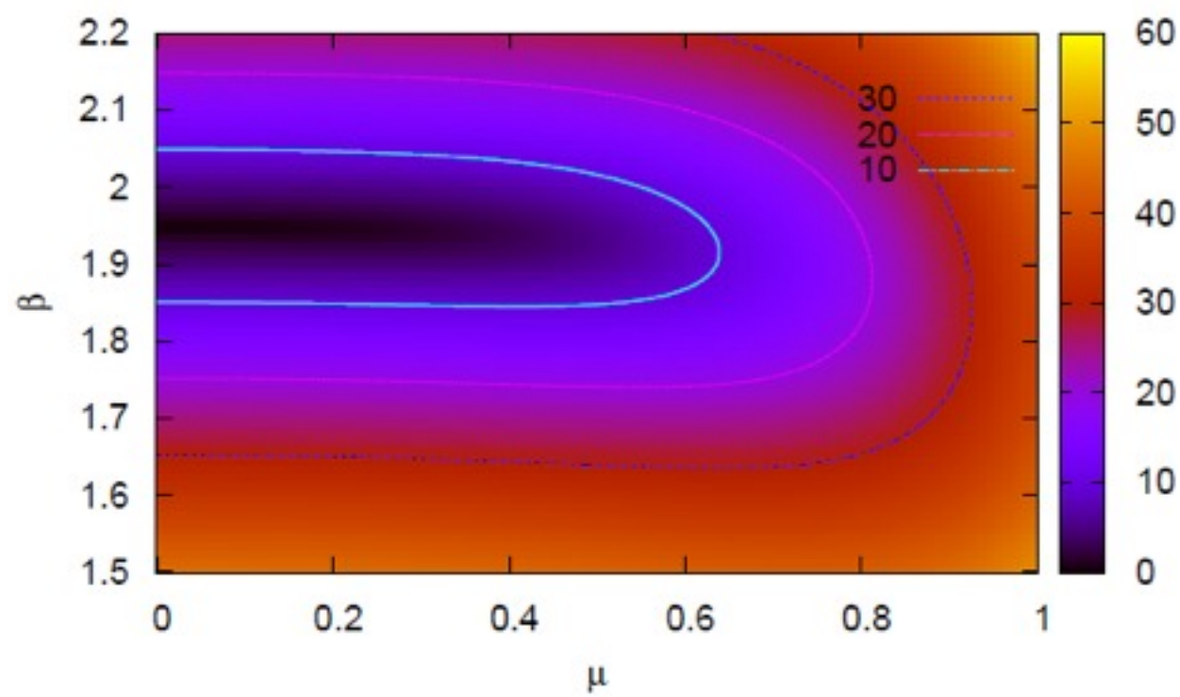
β0=1.85



β0=1.90

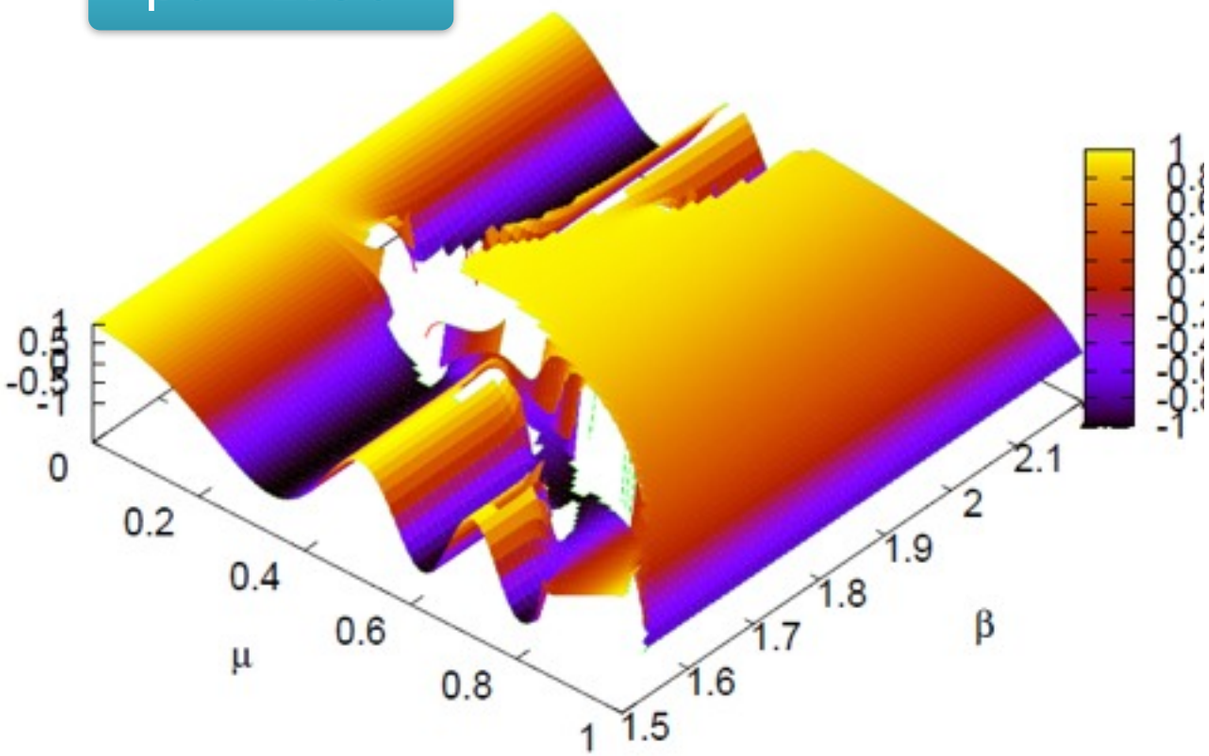


β0=1.95

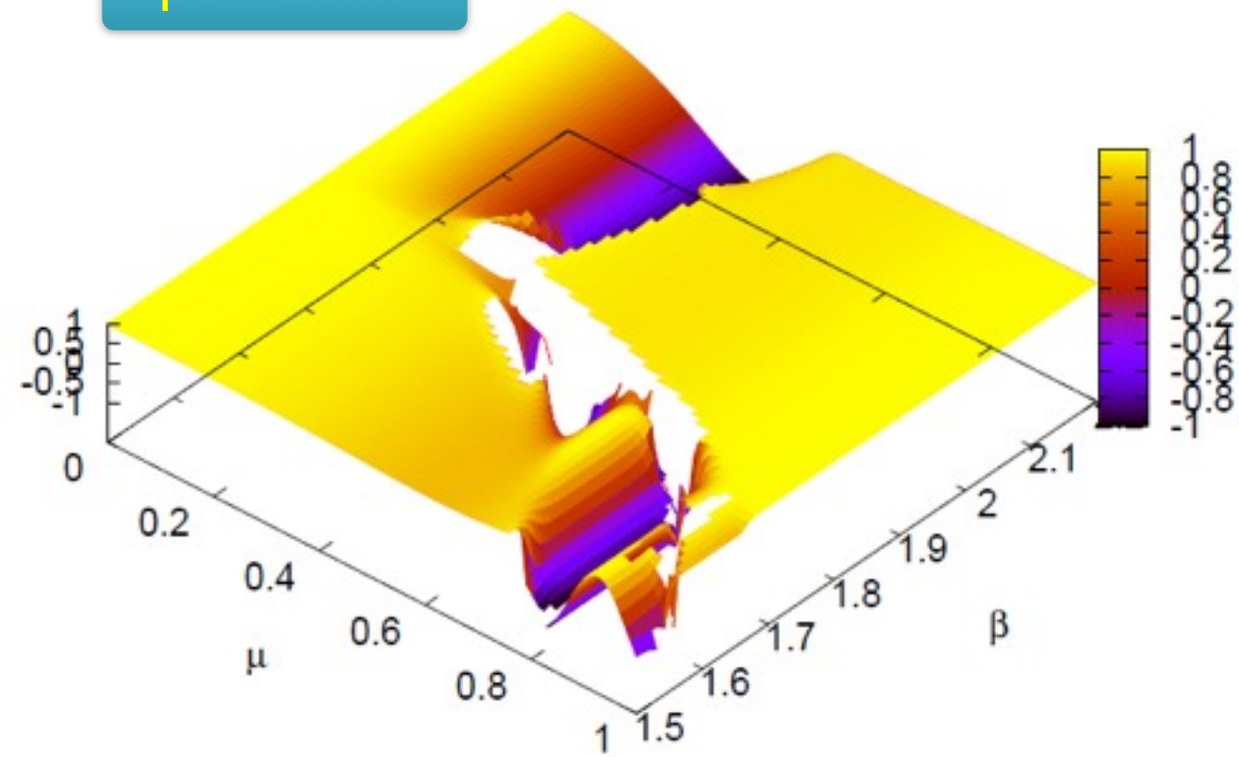


$$\langle \cos 2\theta \rangle = \langle \cos 2\theta \;\; R.F \rangle_0 / \langle R.F \rangle_0$$

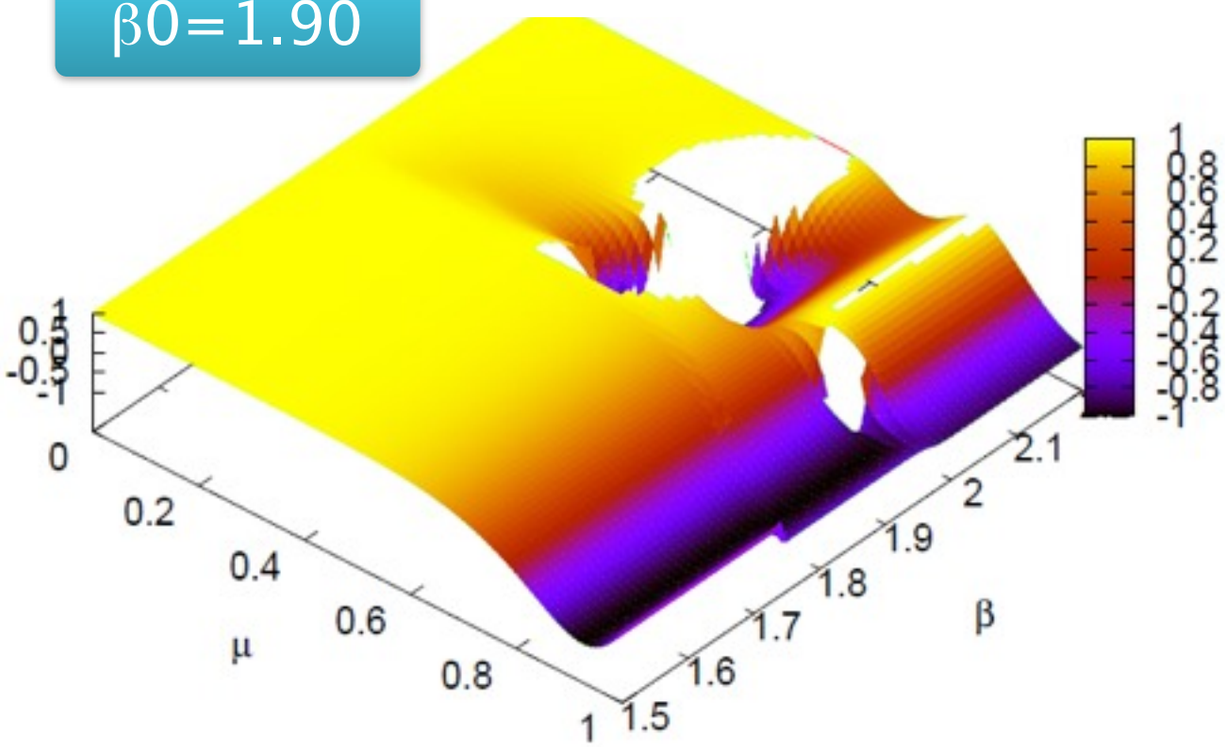
β0=1.80



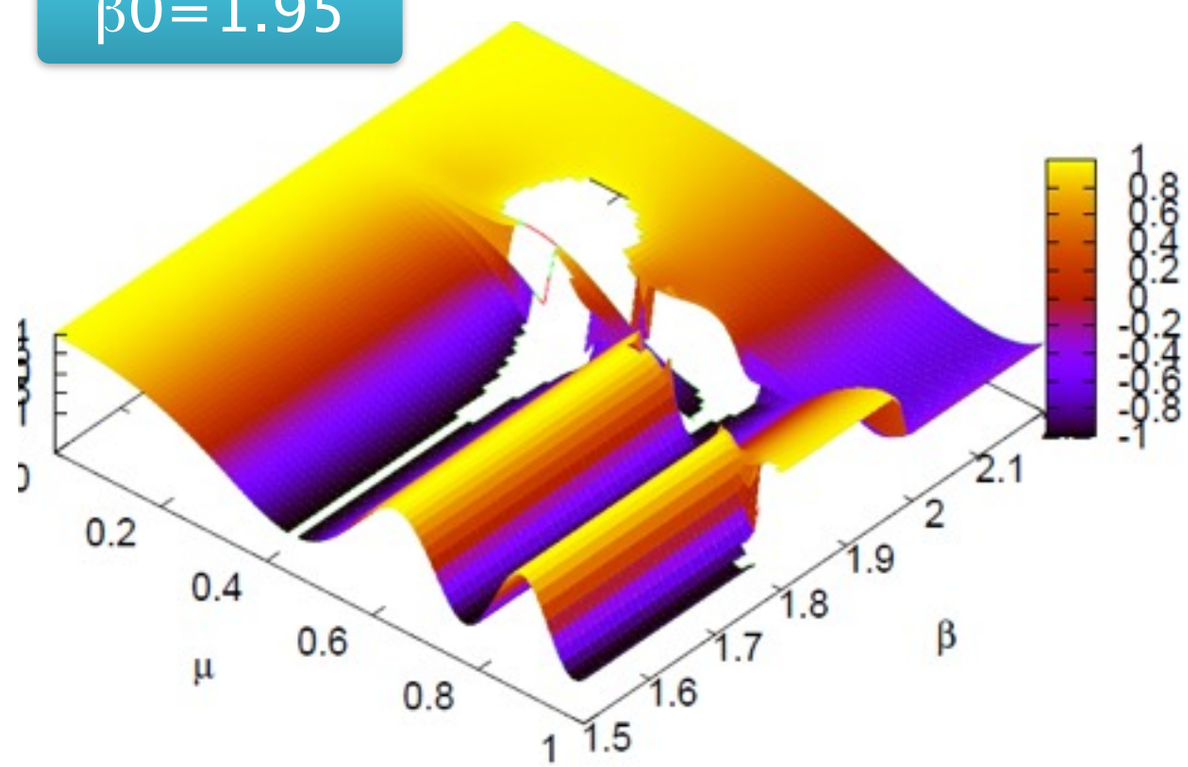
β0=1.85



β0=1.90



β0=1.95



Imaginary Chemical Potential

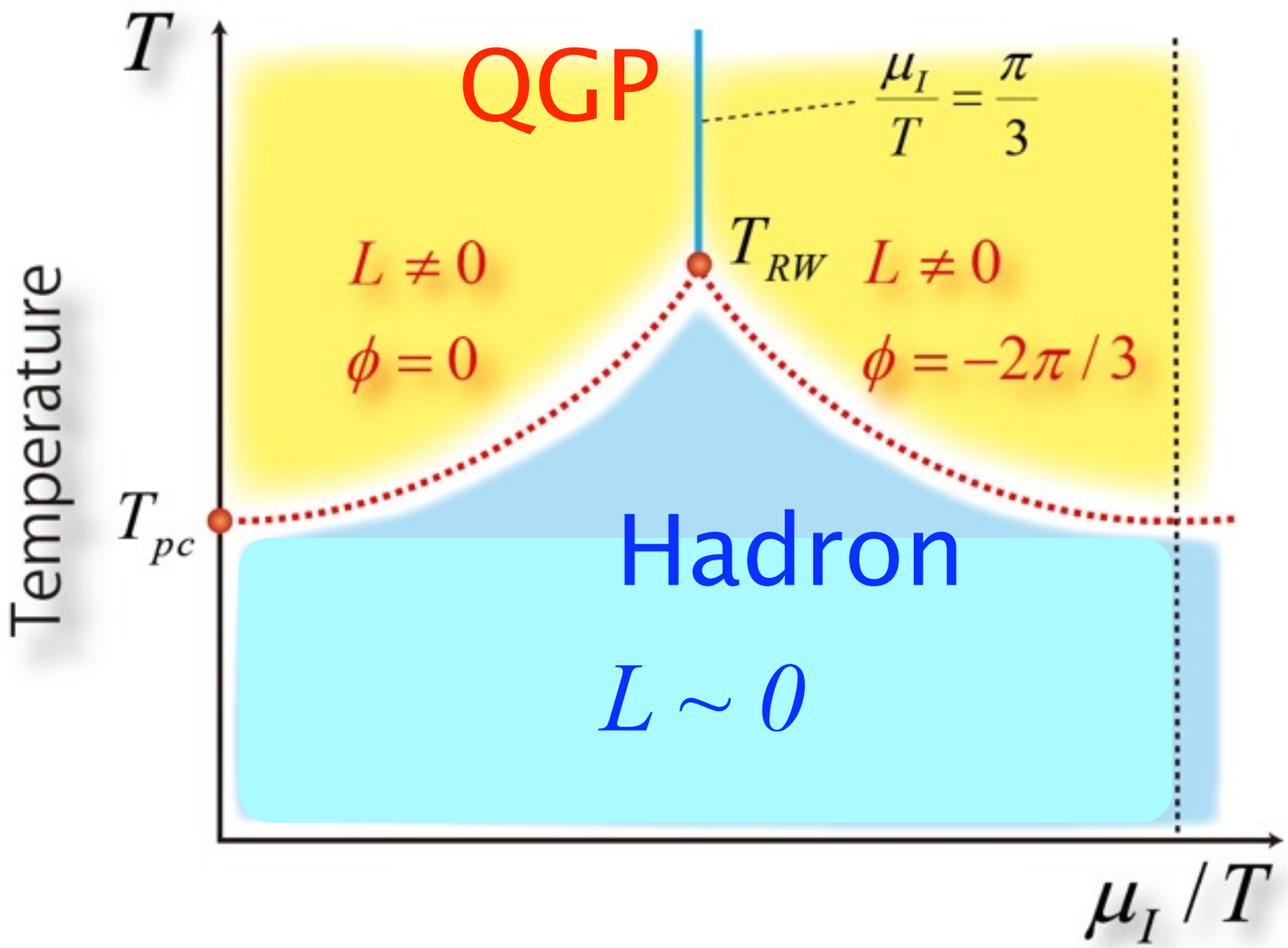
Let's Explore Imaginary World !

1. Introduction
2. Reduction Formula
3. Imainary Chemical Potential

Expected

Phase diagram in μI regions

a la Reberge-Weiss



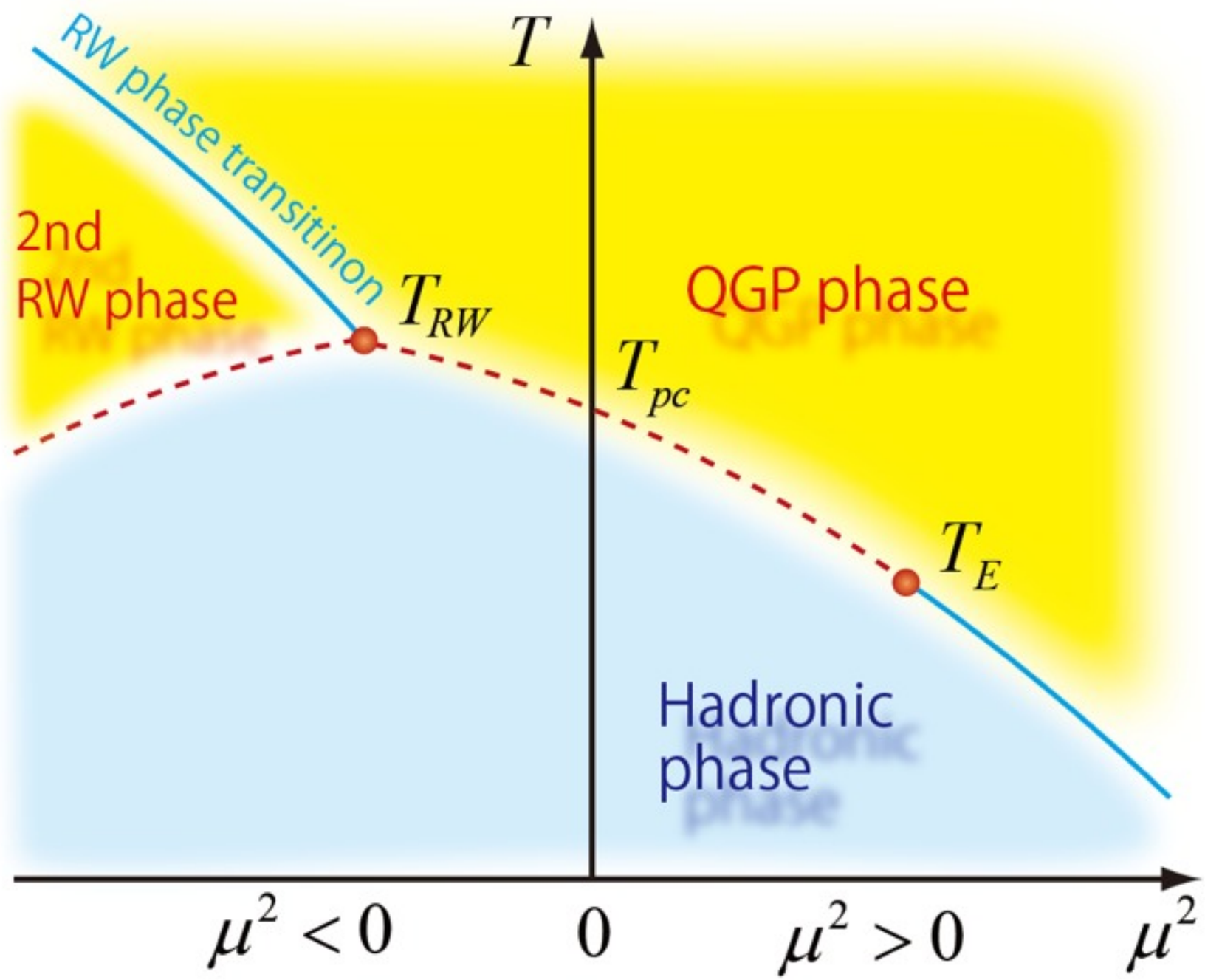
Polyakov loop

$$P = L_P \exp(i\phi_P)$$

If μ is pure imaginary

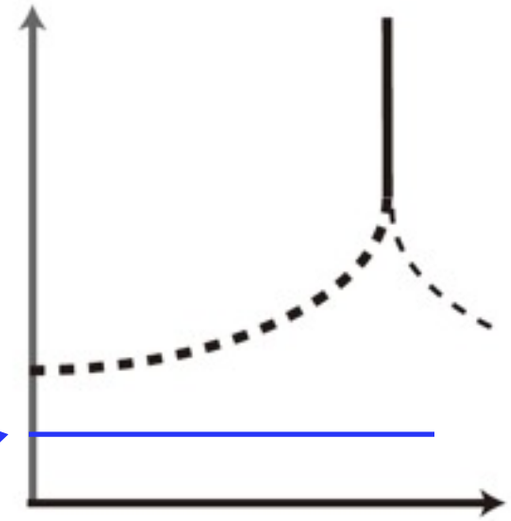
there is no sign problem.

Imaginary to real chemical potential

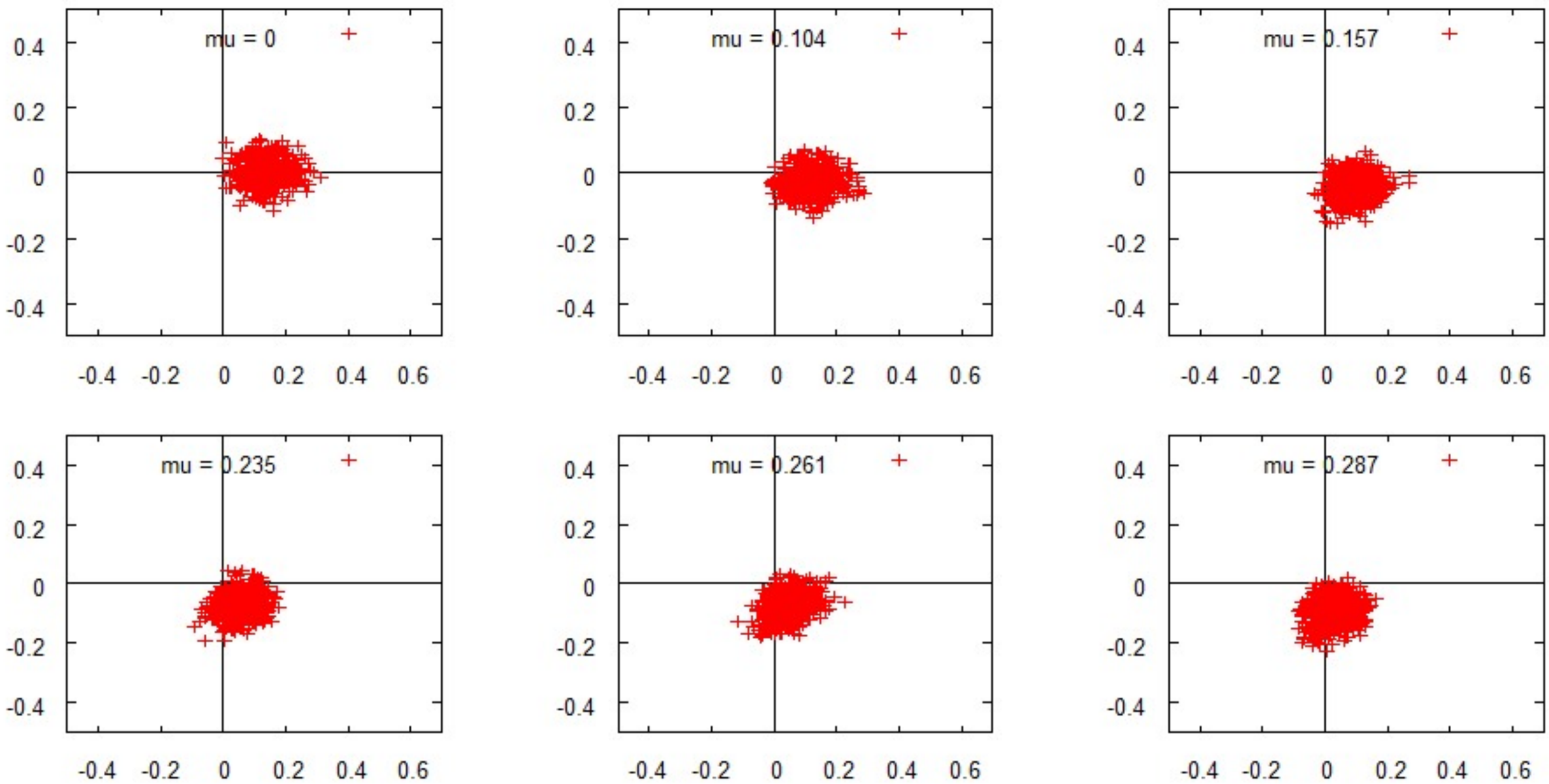


Polyakov loop scatter plot

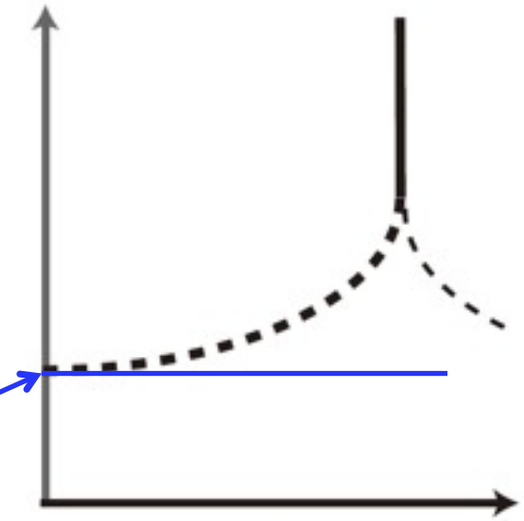
T=0.93Tc
beta = 1.80



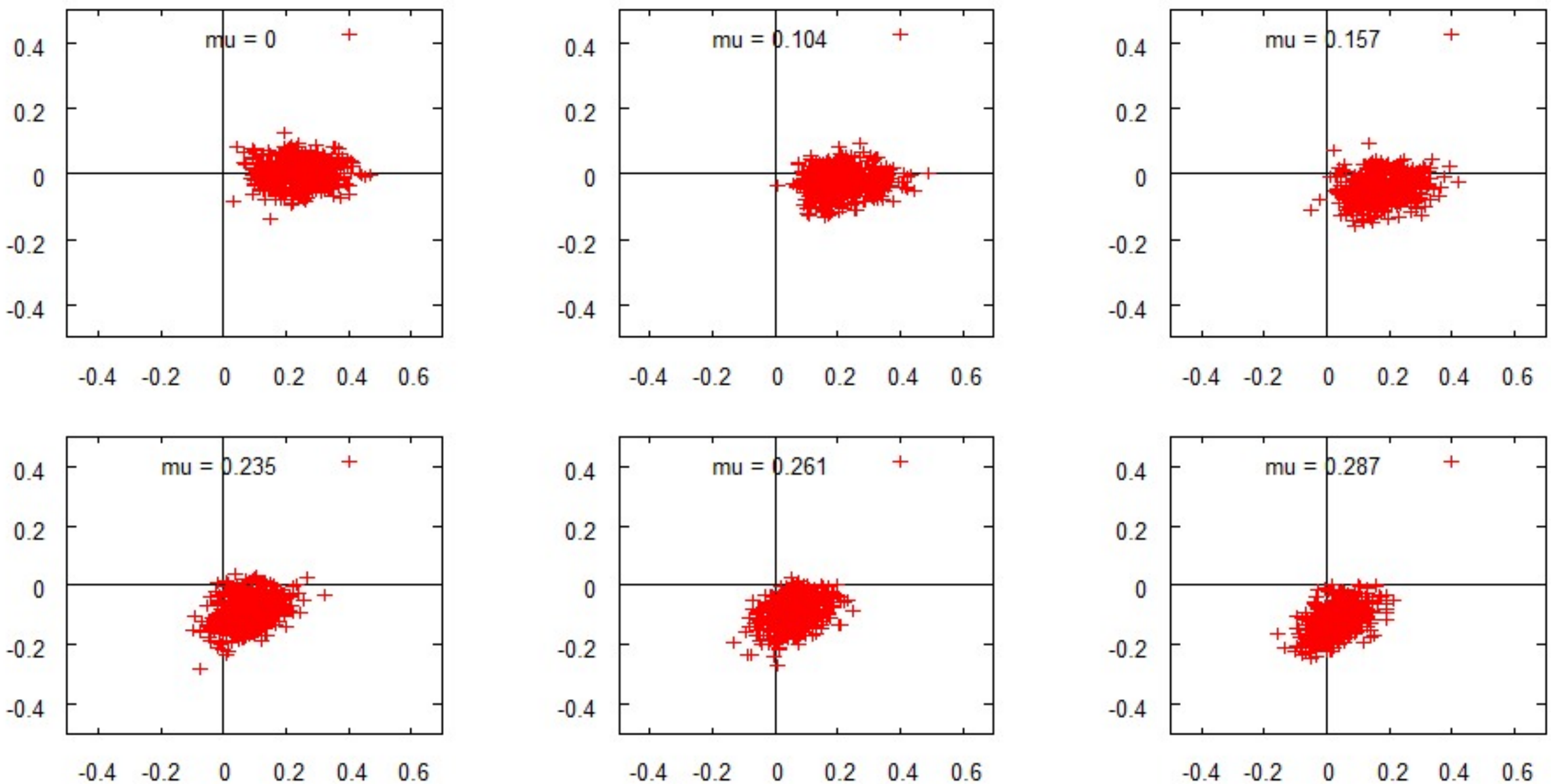
ϕ changes continously



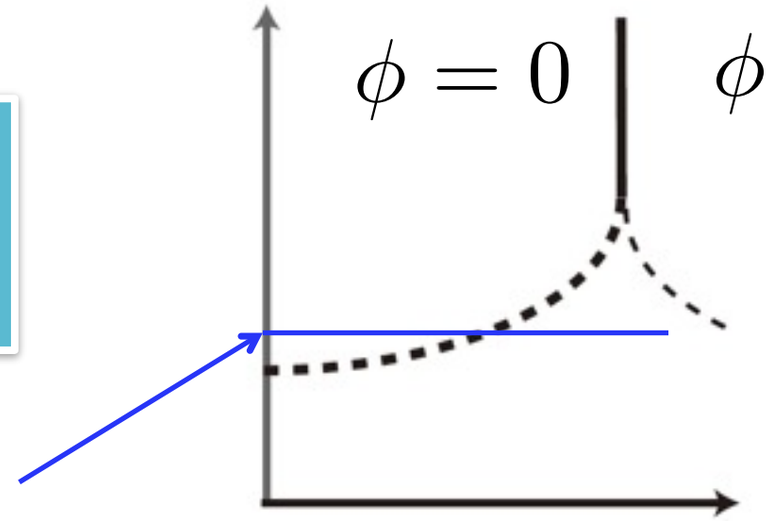
T=0.99Tc
beta = 1.85



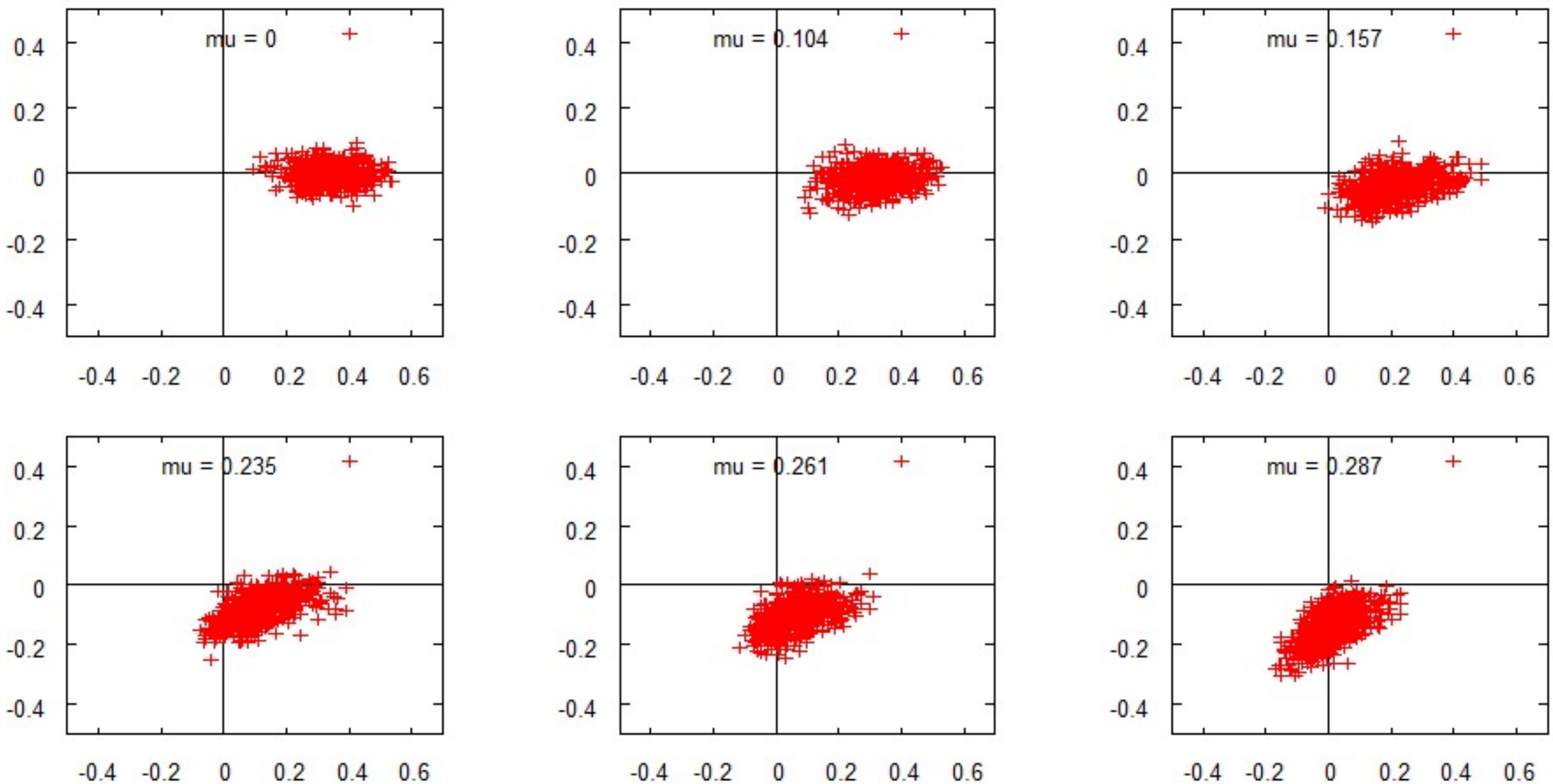
ϕ changes continously



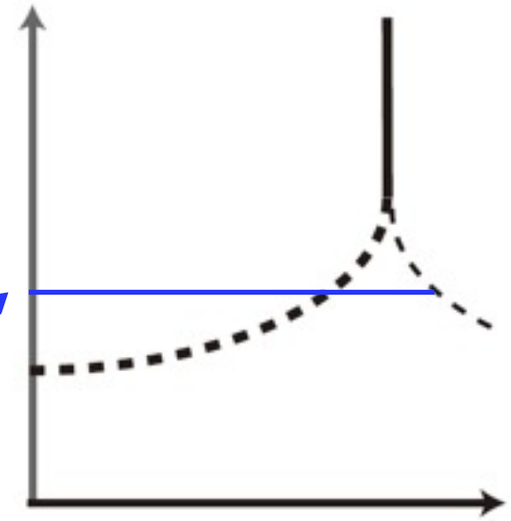
T=1.02Tc
beta = 1.87



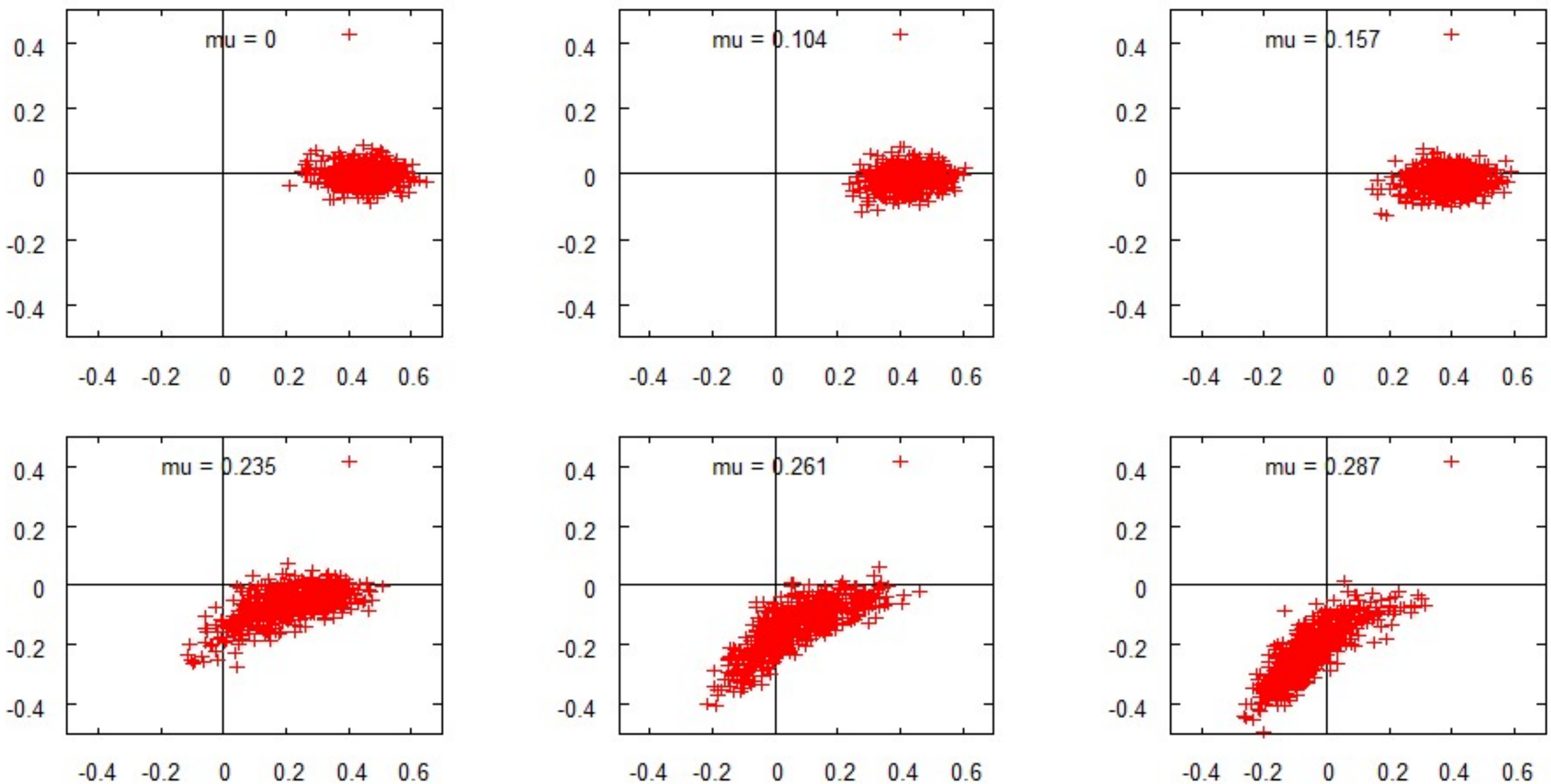
$\phi = 0$ for $\mu<0.1$
finite for $\mu>0.1$



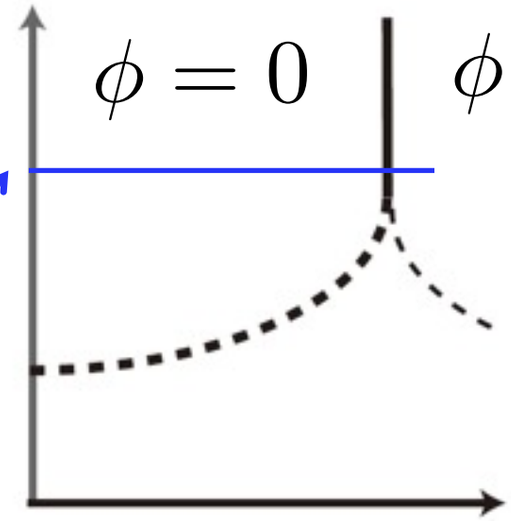
T=1.08Tc
beta = 1.90



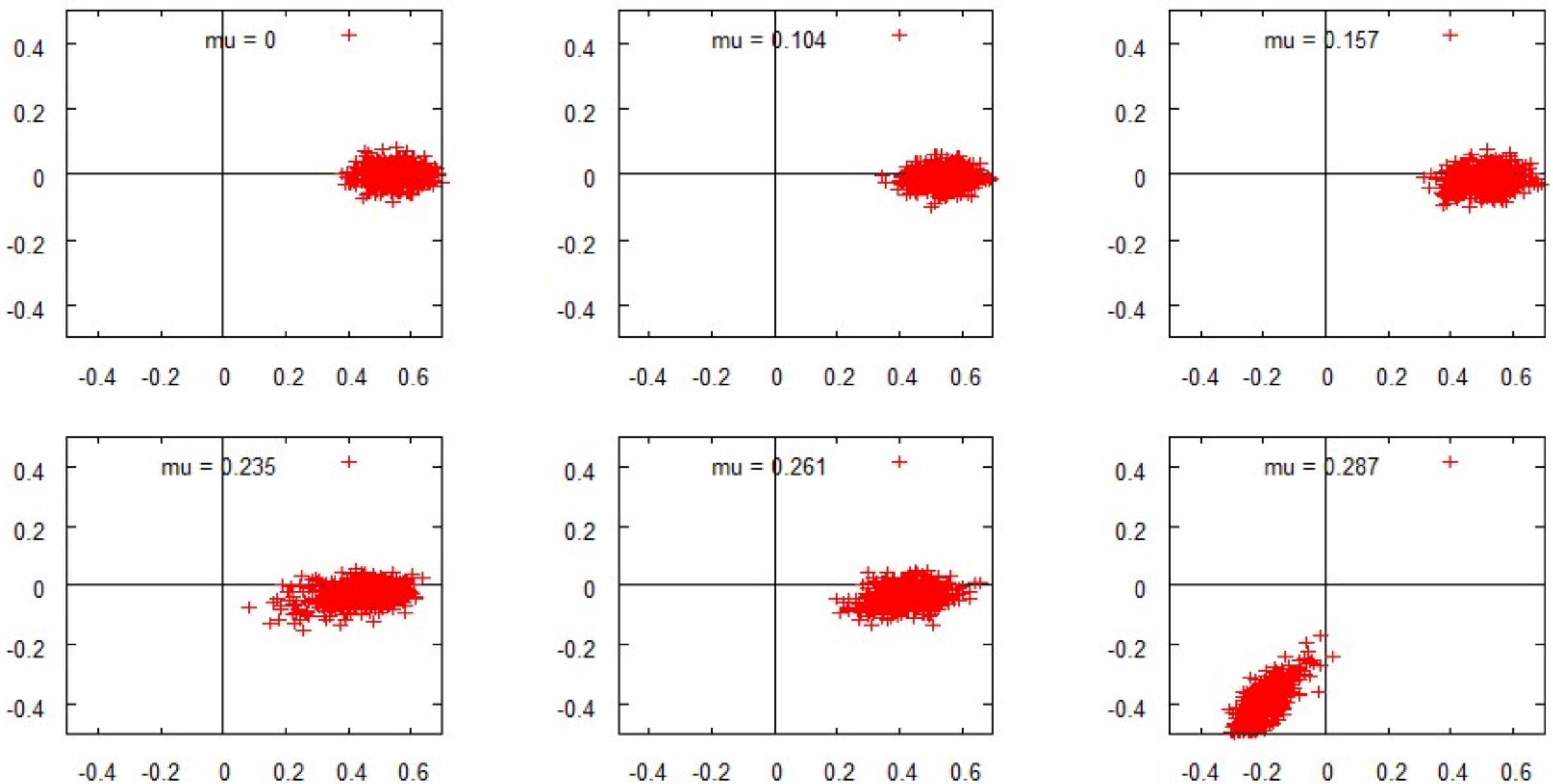
$\phi = 0$ for $\mu<0.2$
finite for $\mu>0.2$



T=1.20Tc
beta = 1.95

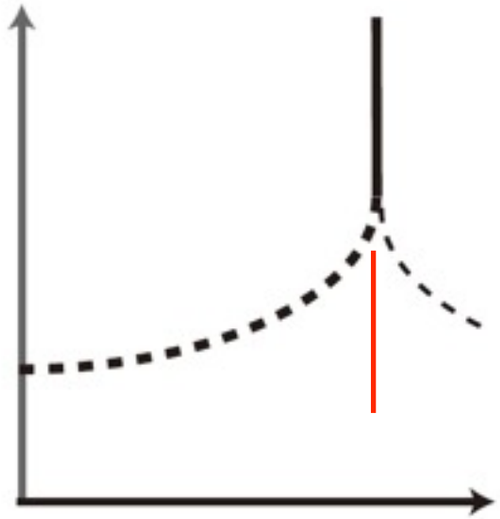


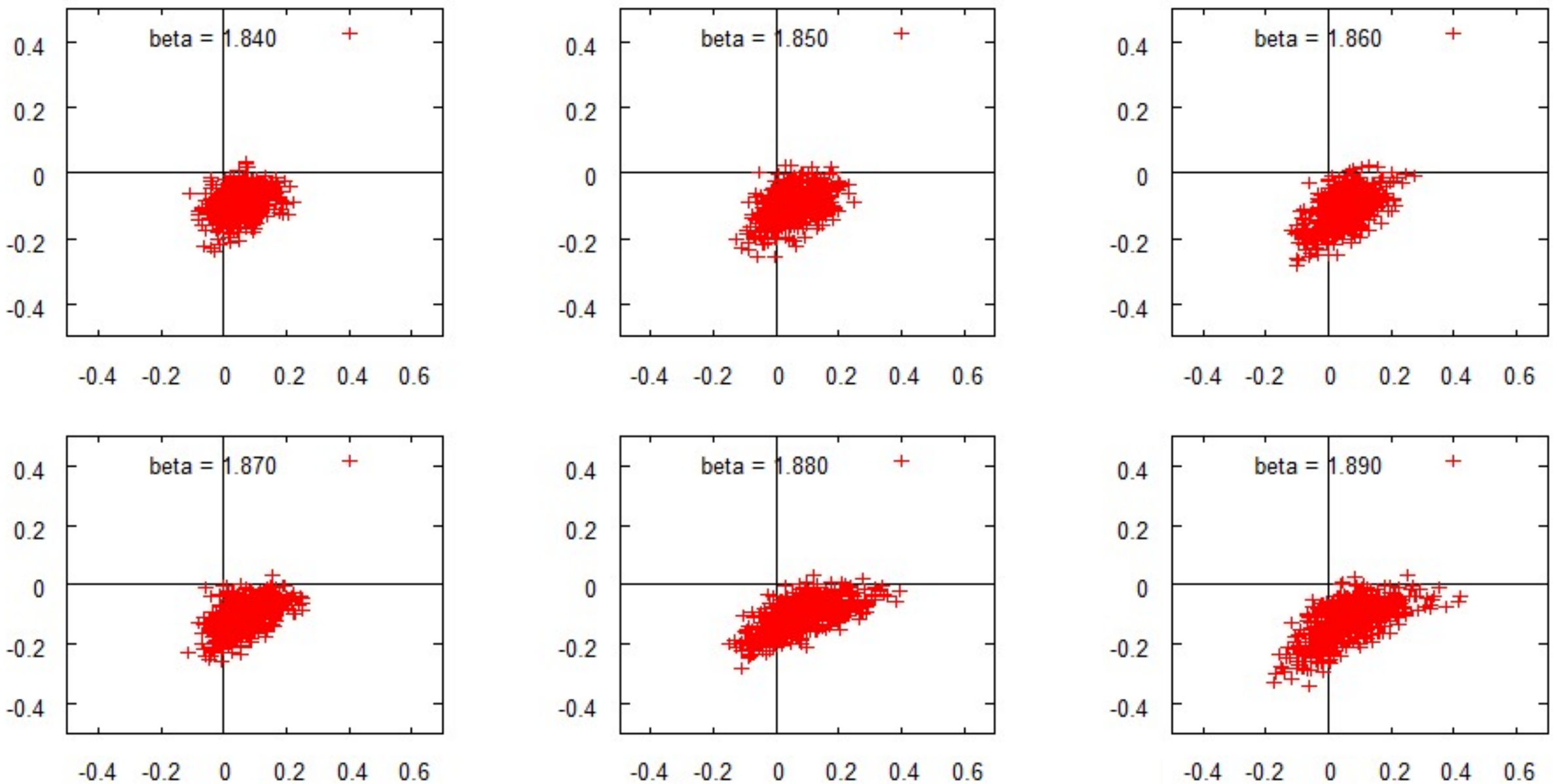
ϕ jumps at μ=0.26



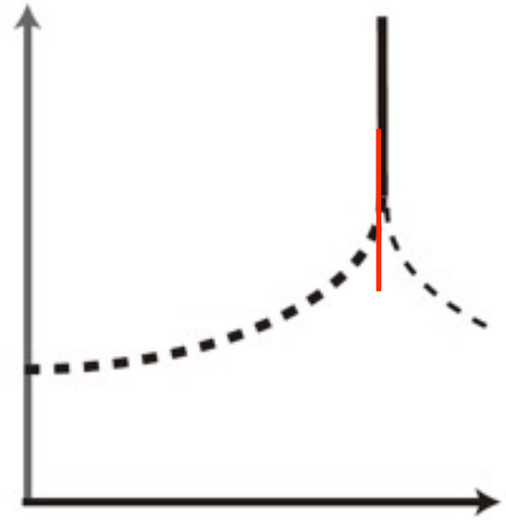
Polyakov loop scatter plot

$\beta = 1.84\text{–}1.89$

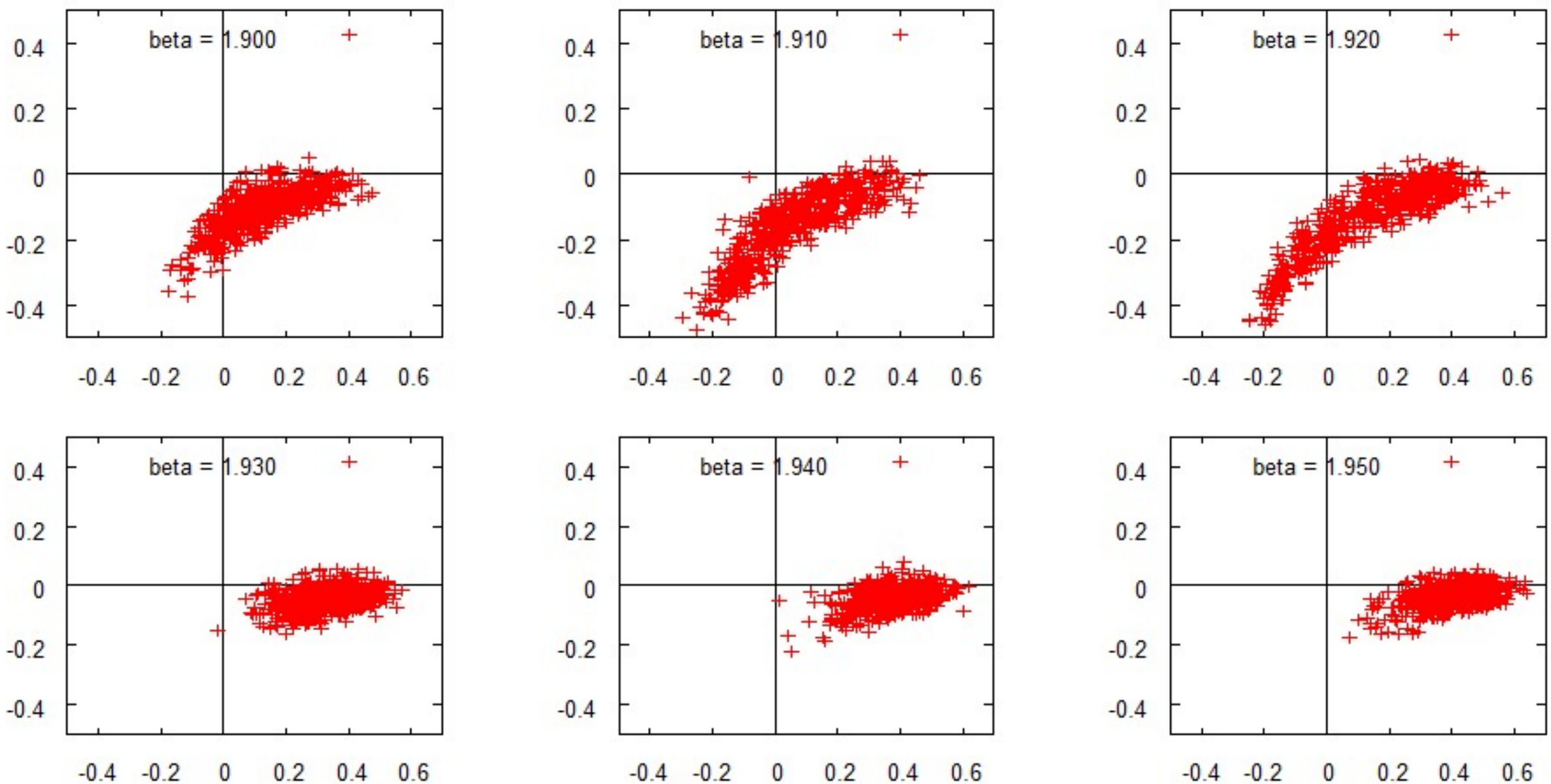




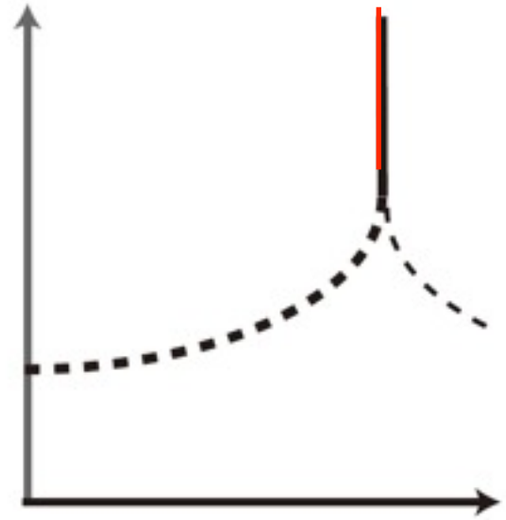
β = 1.90–1.95



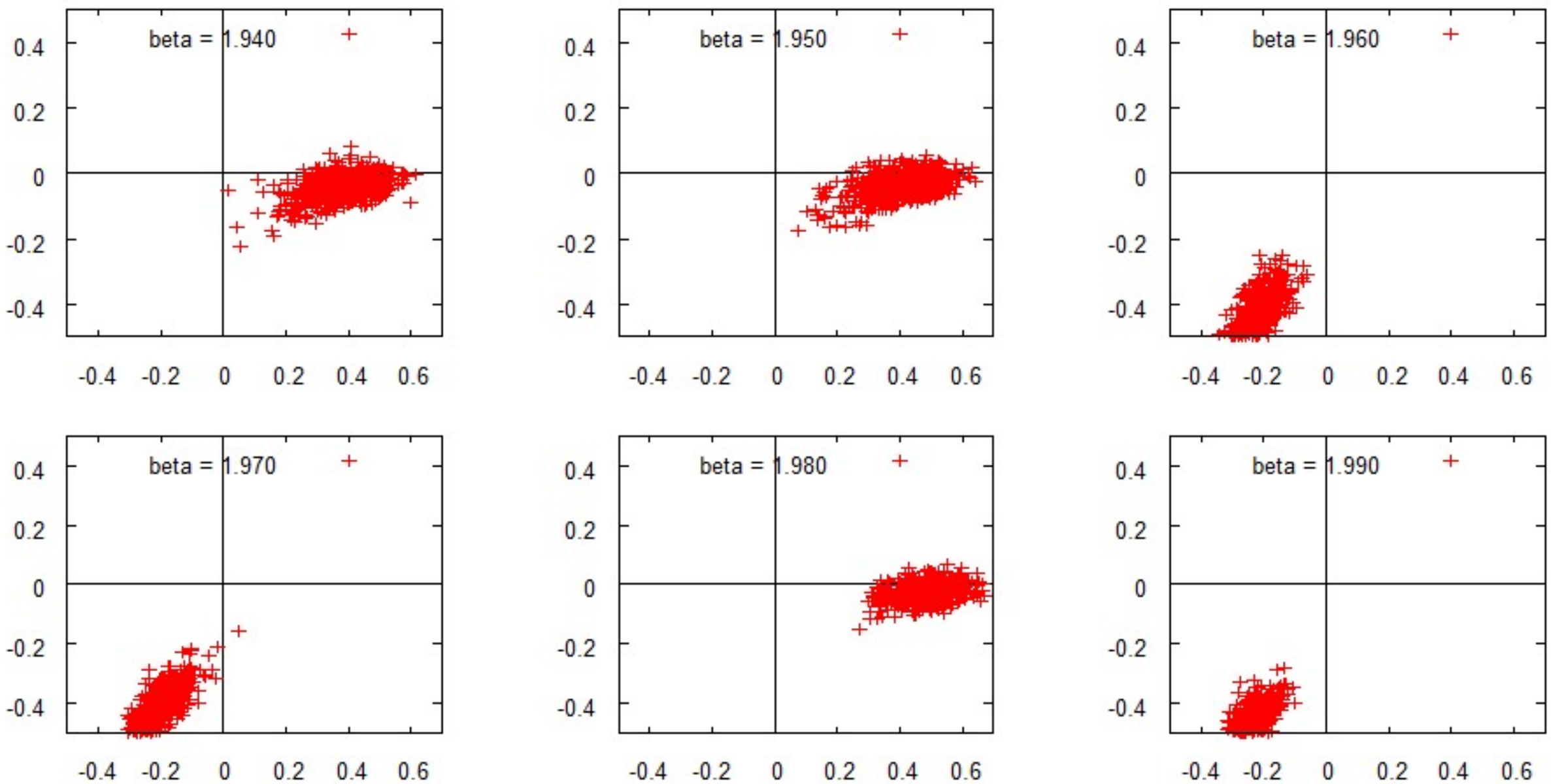
2-peak behaviour around β=1.92



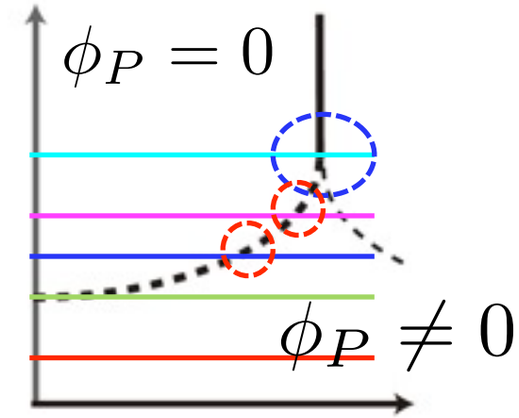
$\beta = 1.94\text{–}1.99$

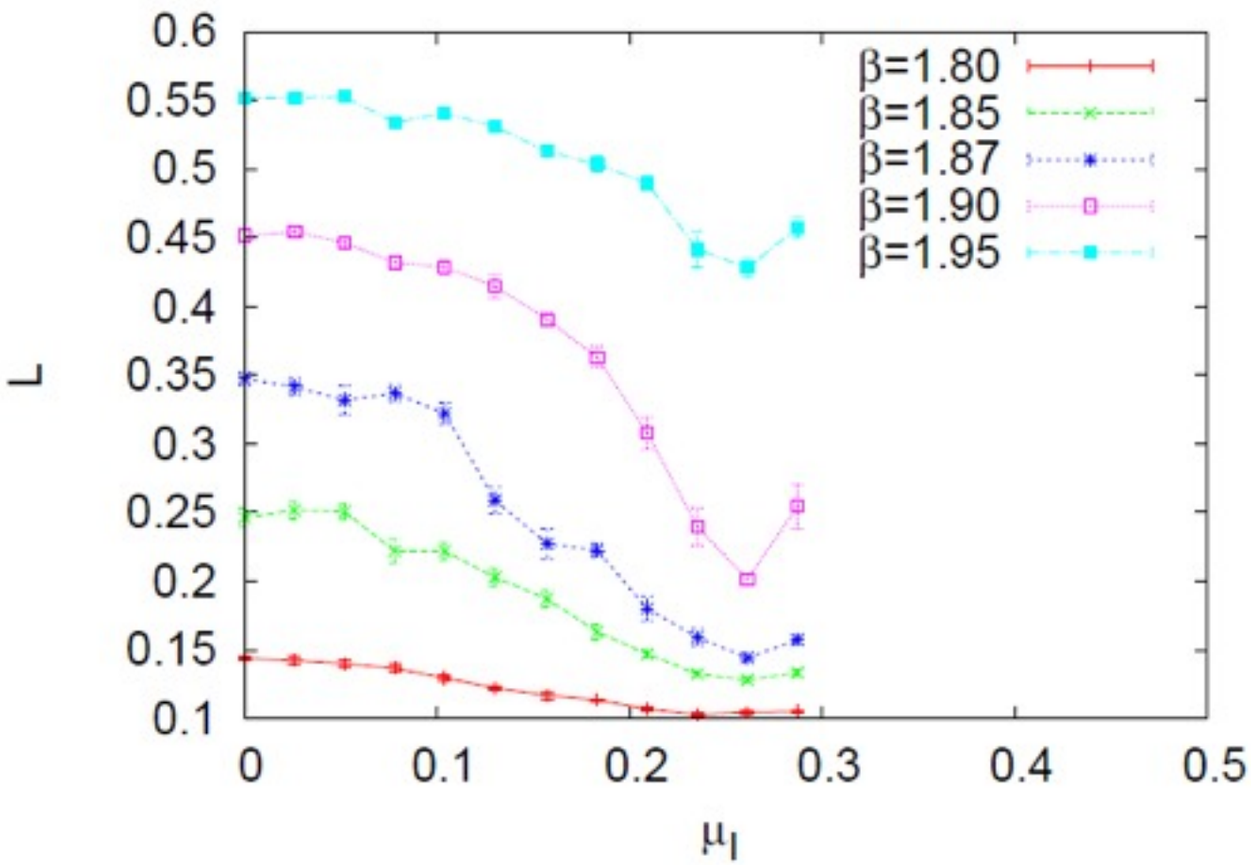


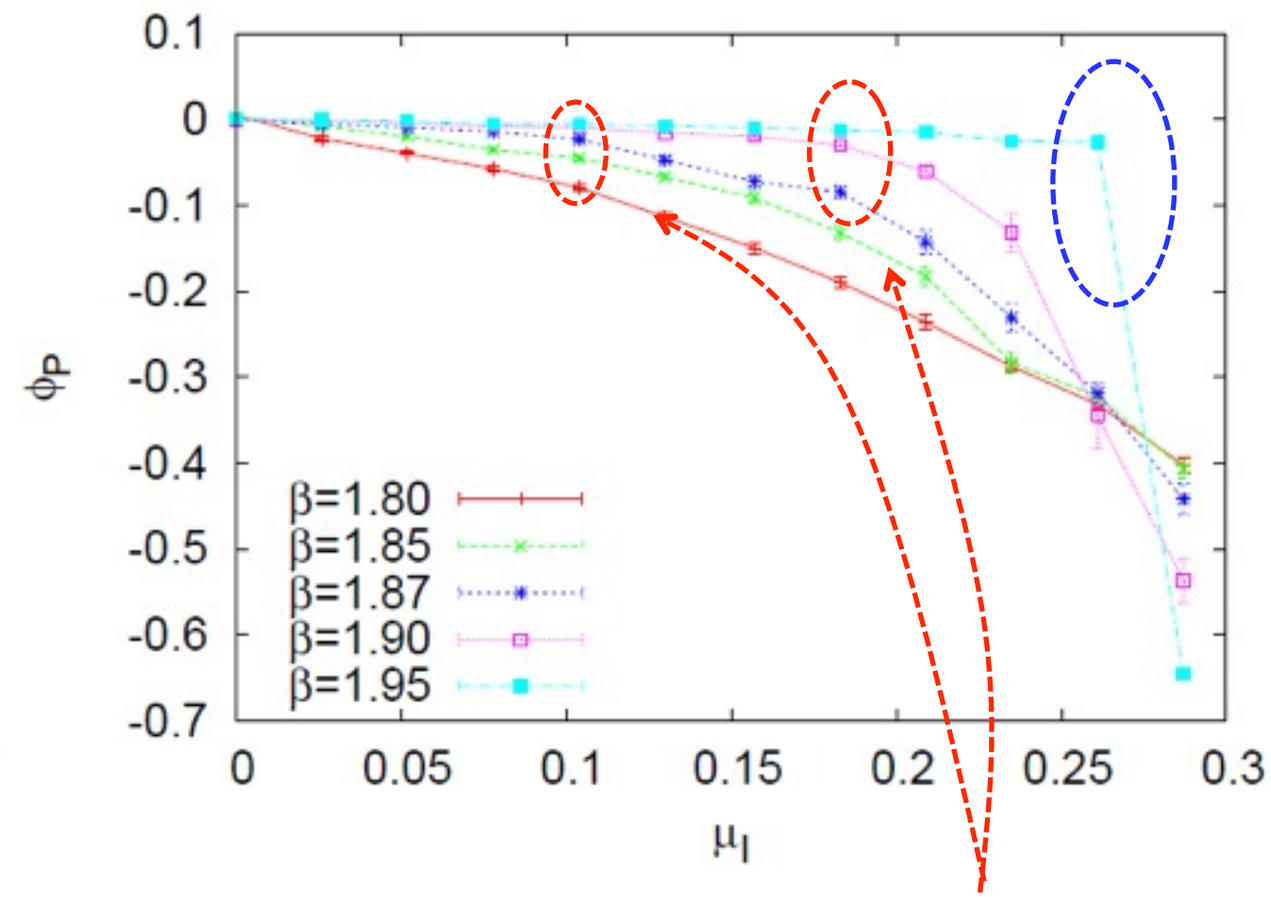
spontanous breaking in high T



Polyakov Loop (L: absolute, r: phase) μ-dependence





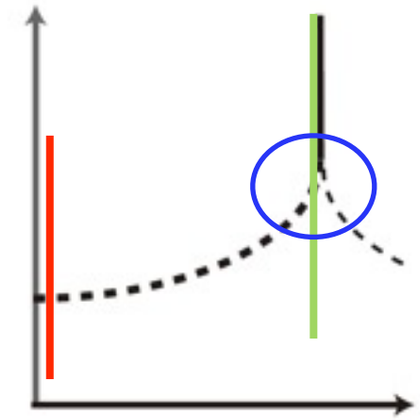


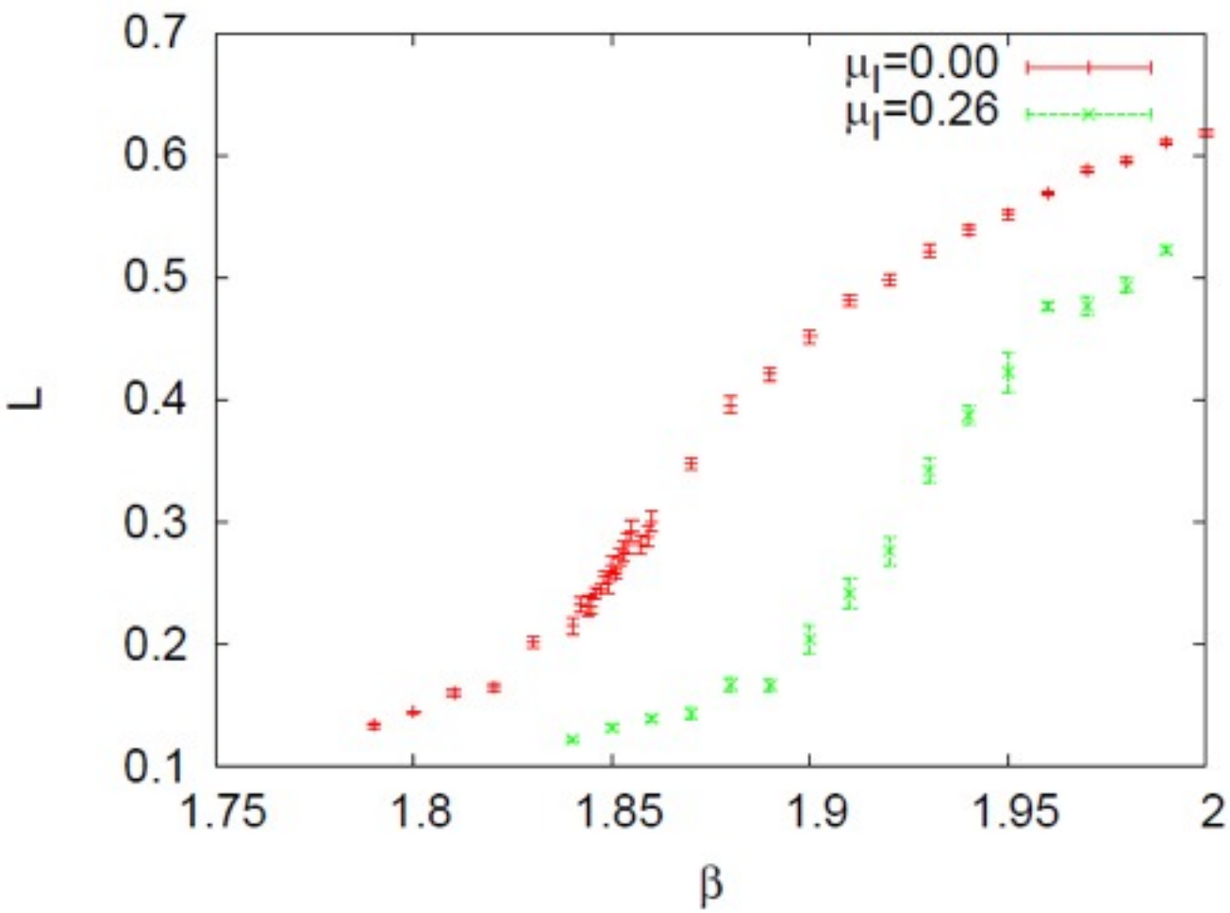
deconfinement transition

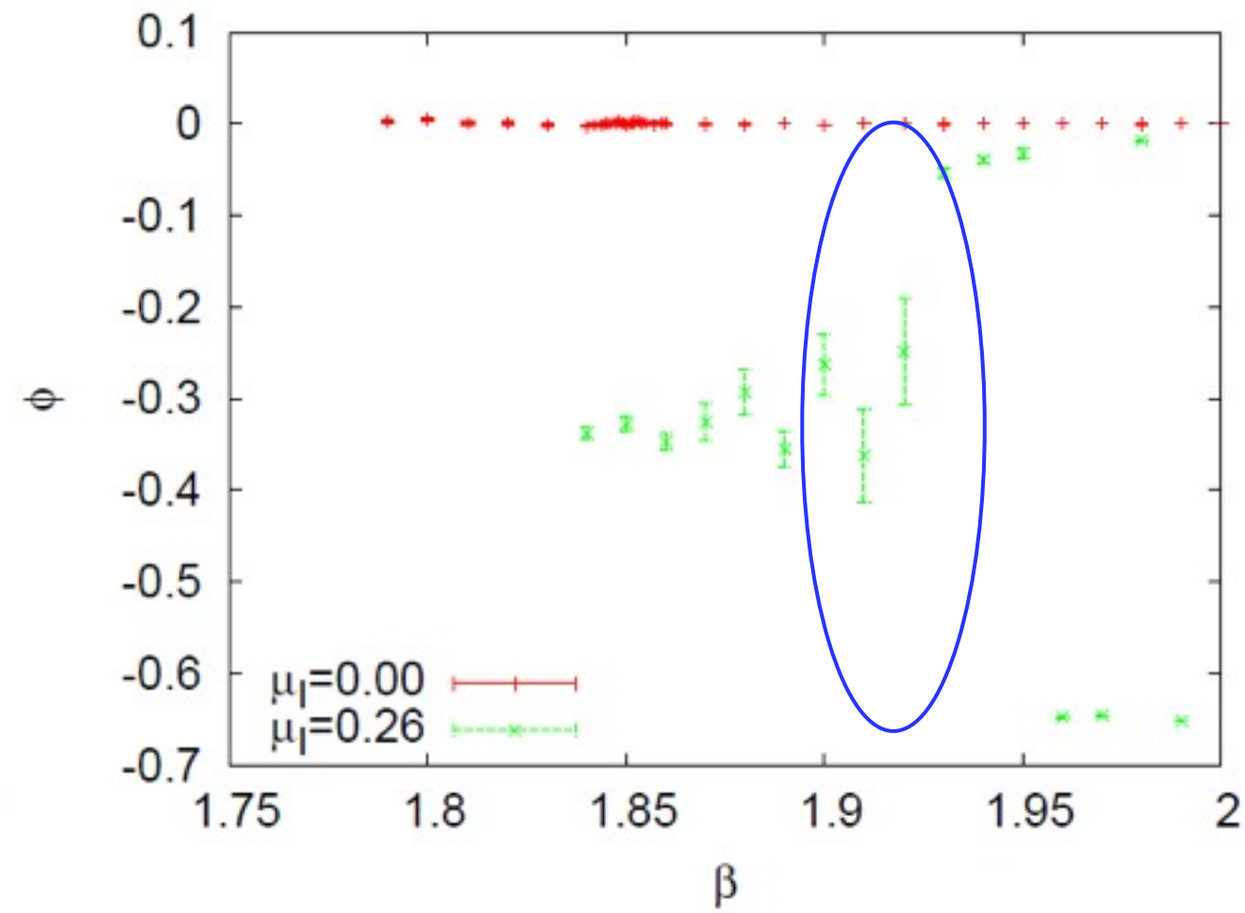
- **We determine the shape of the pseudo-critical line,**
- **assuming the phase of PL is also an order parameter of the deconfinement transition in μI region.**

Polyakov Loop T-dependence

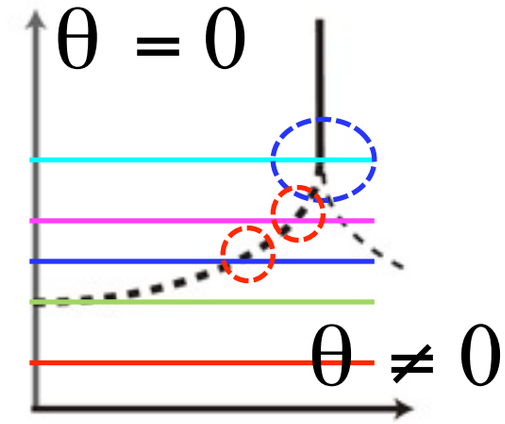
RW End point

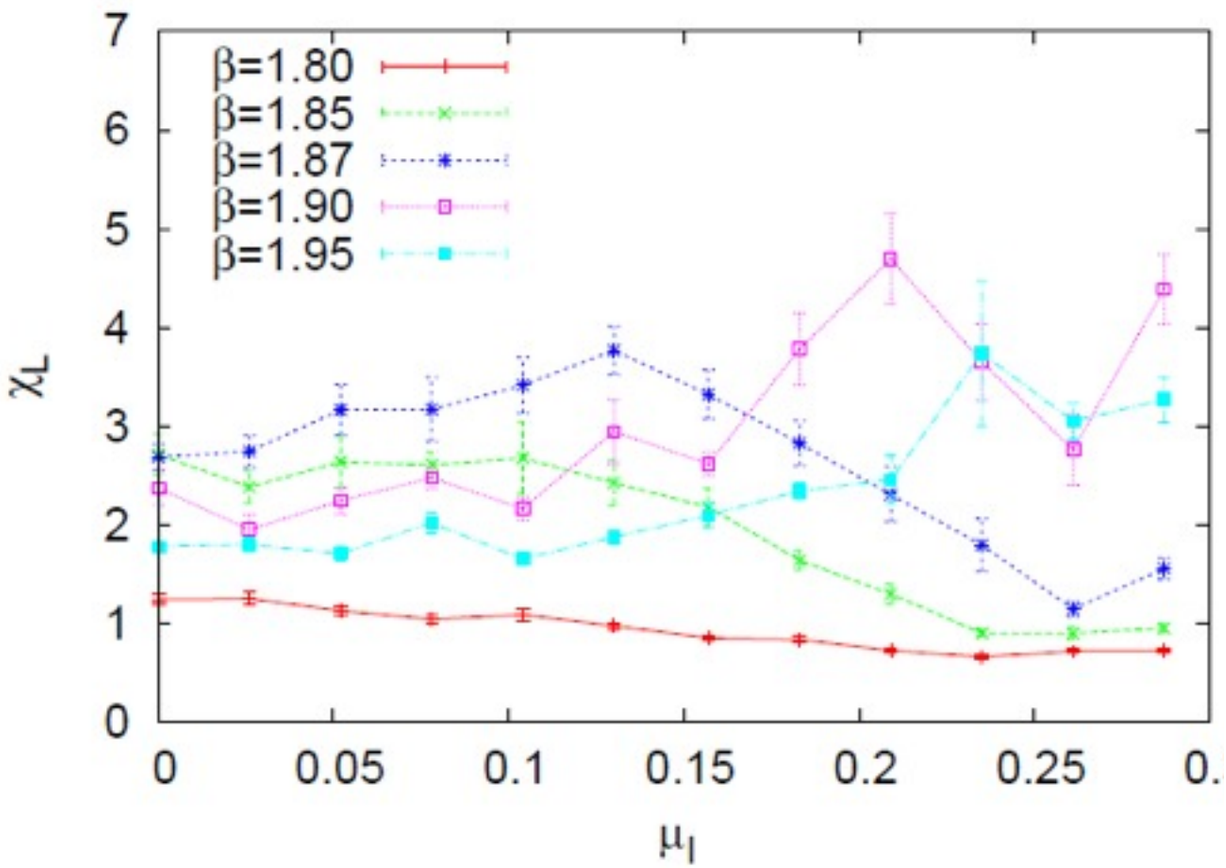


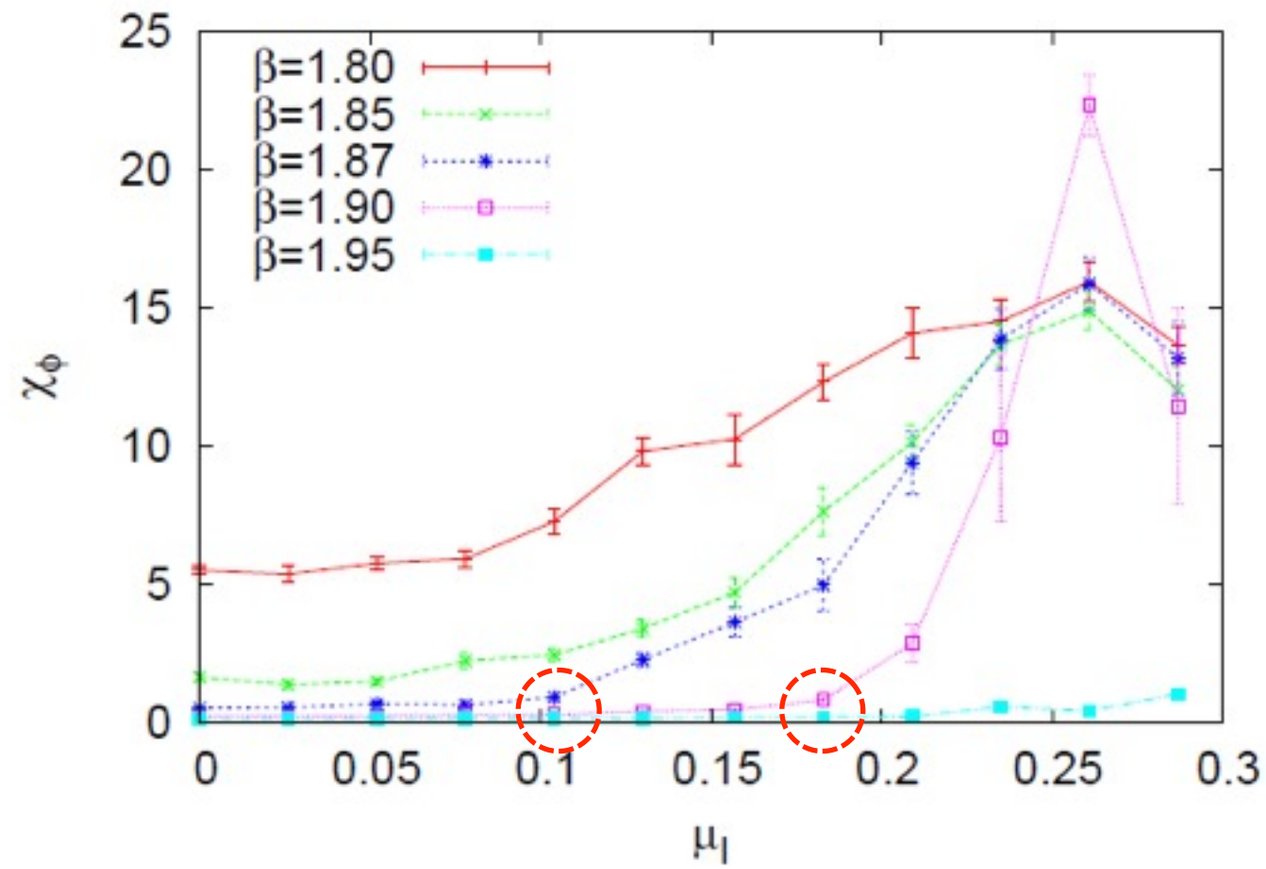




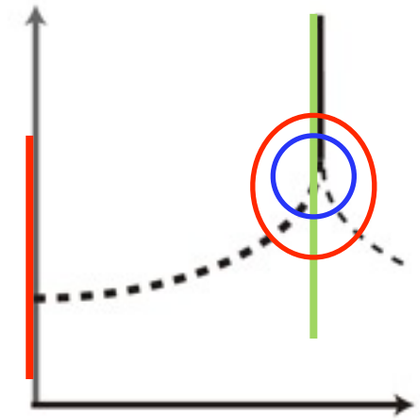
Susceptibility
μ-dependence

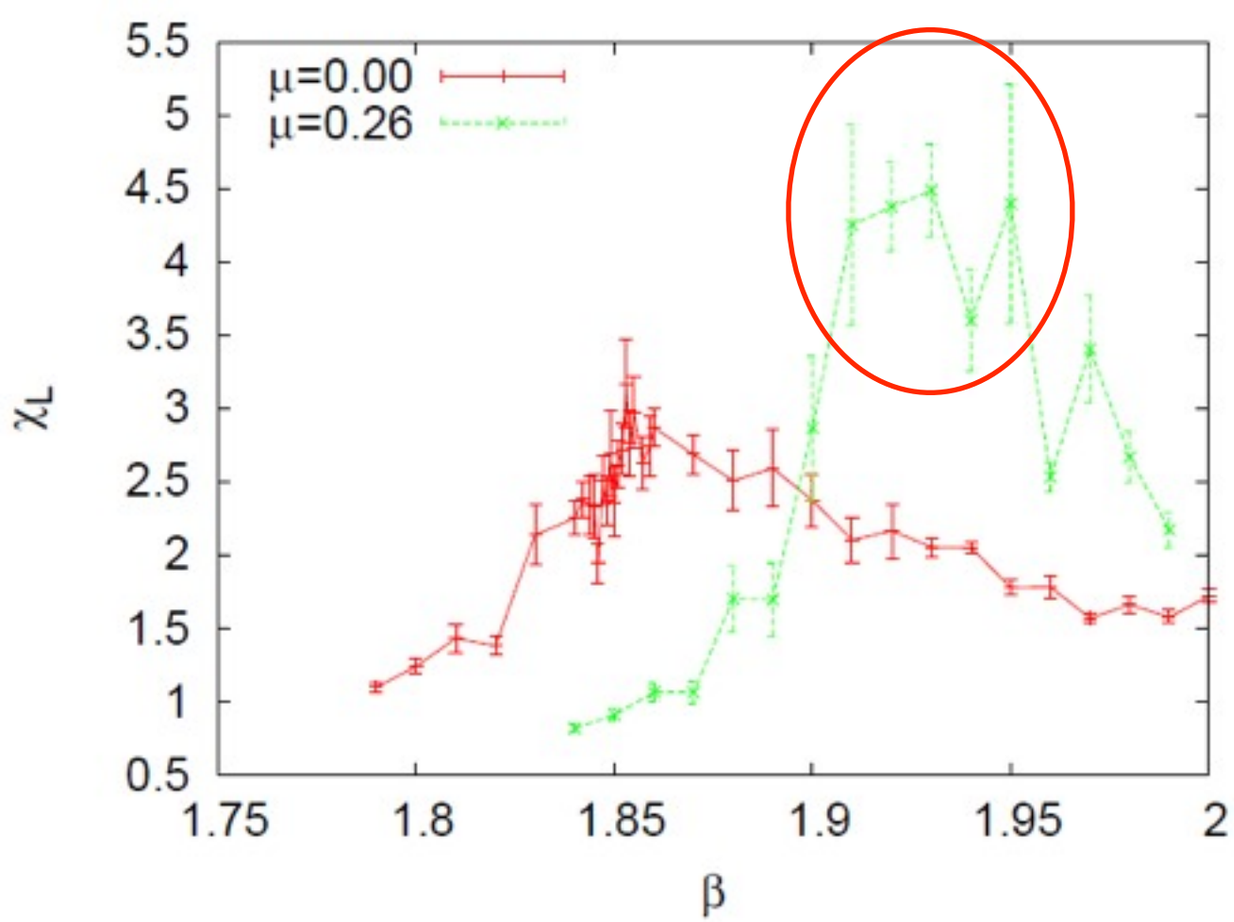


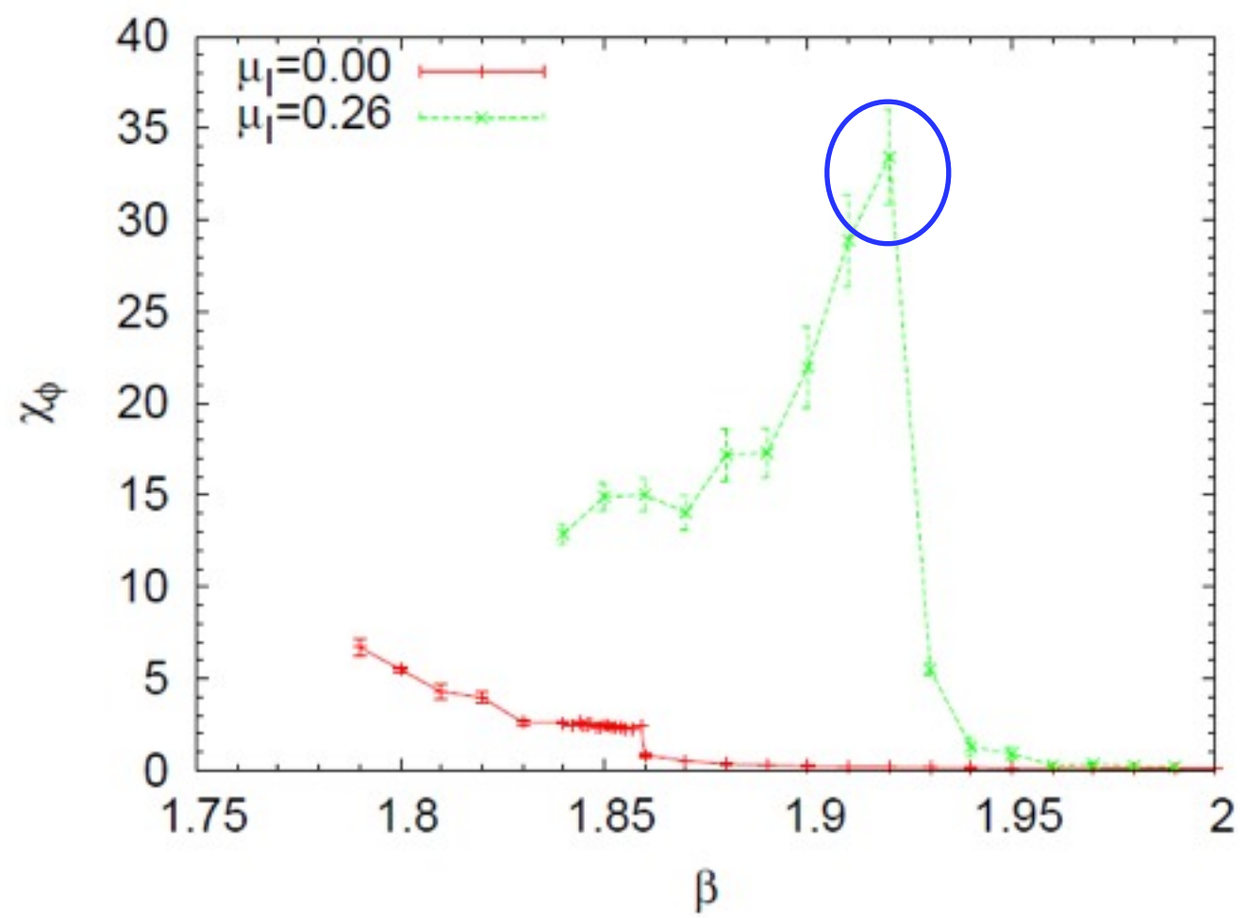




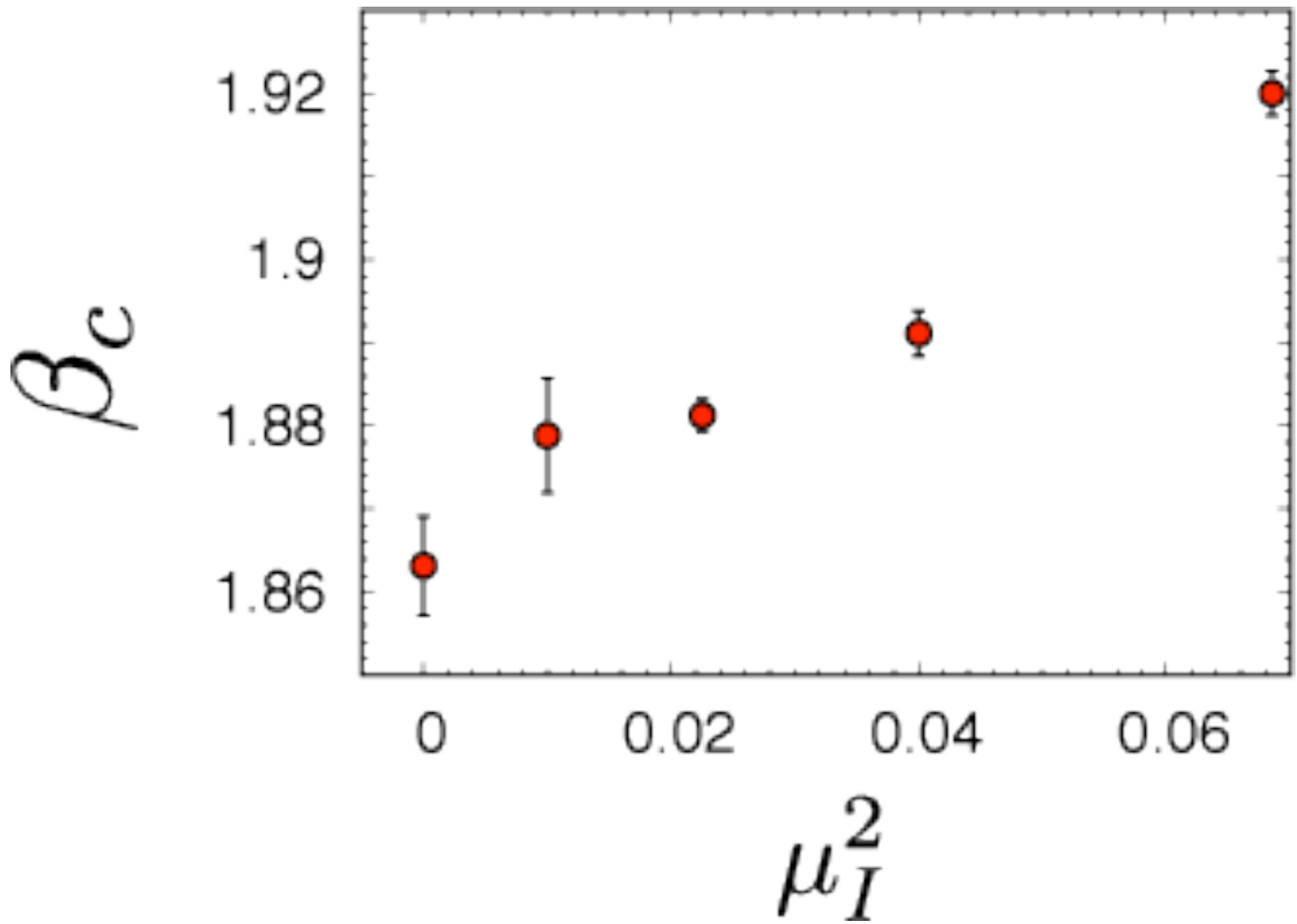
Susceptibility T-dependence





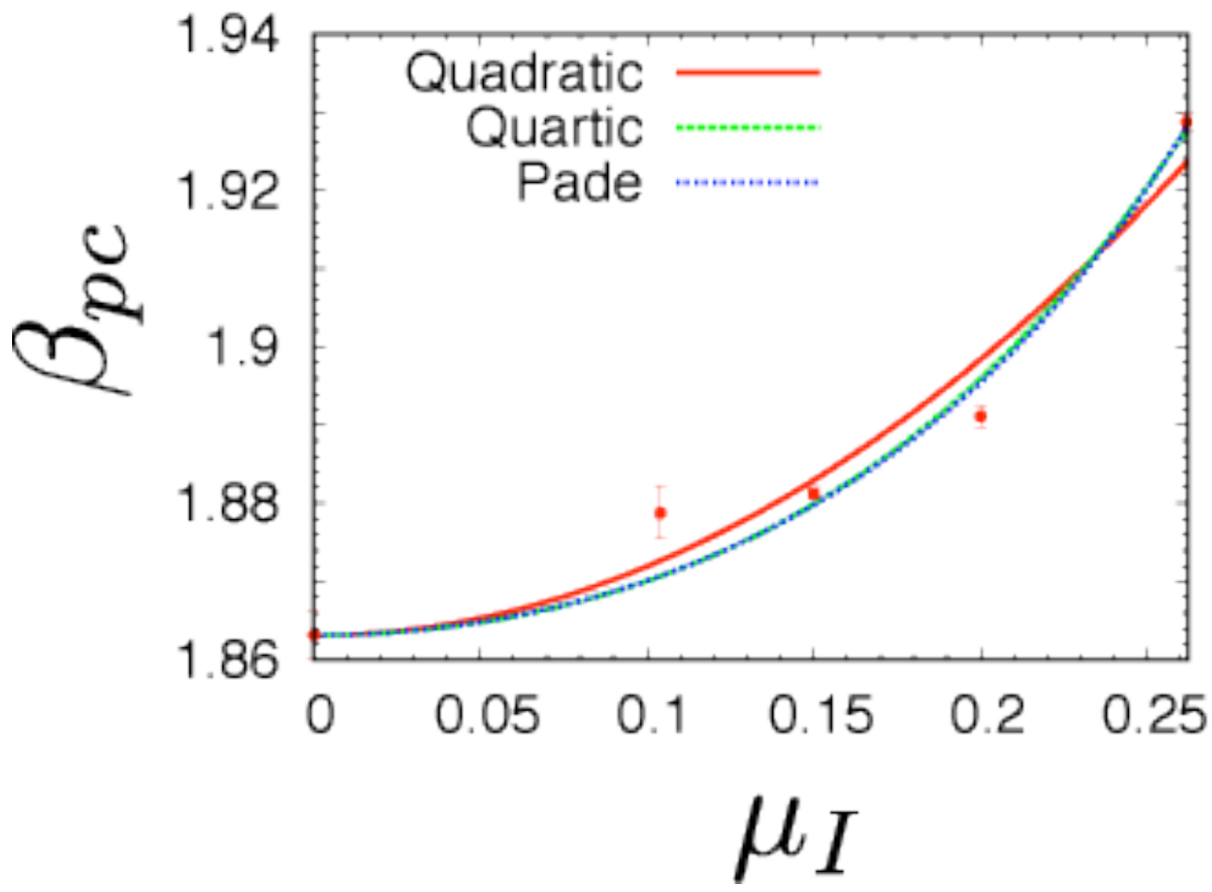


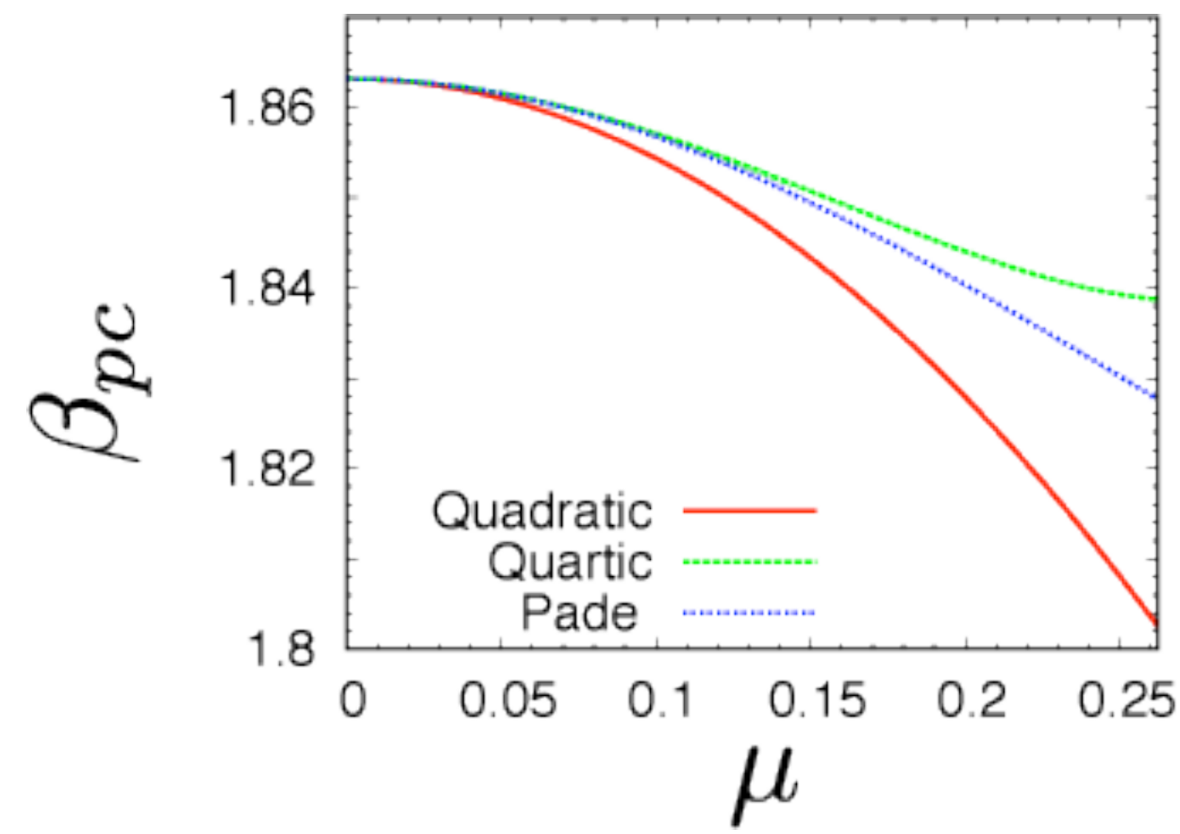
- **Deconfinement and RW transition lines end at the same point within errors.**
- **The RW end-point shows the 2nd order behavior.**



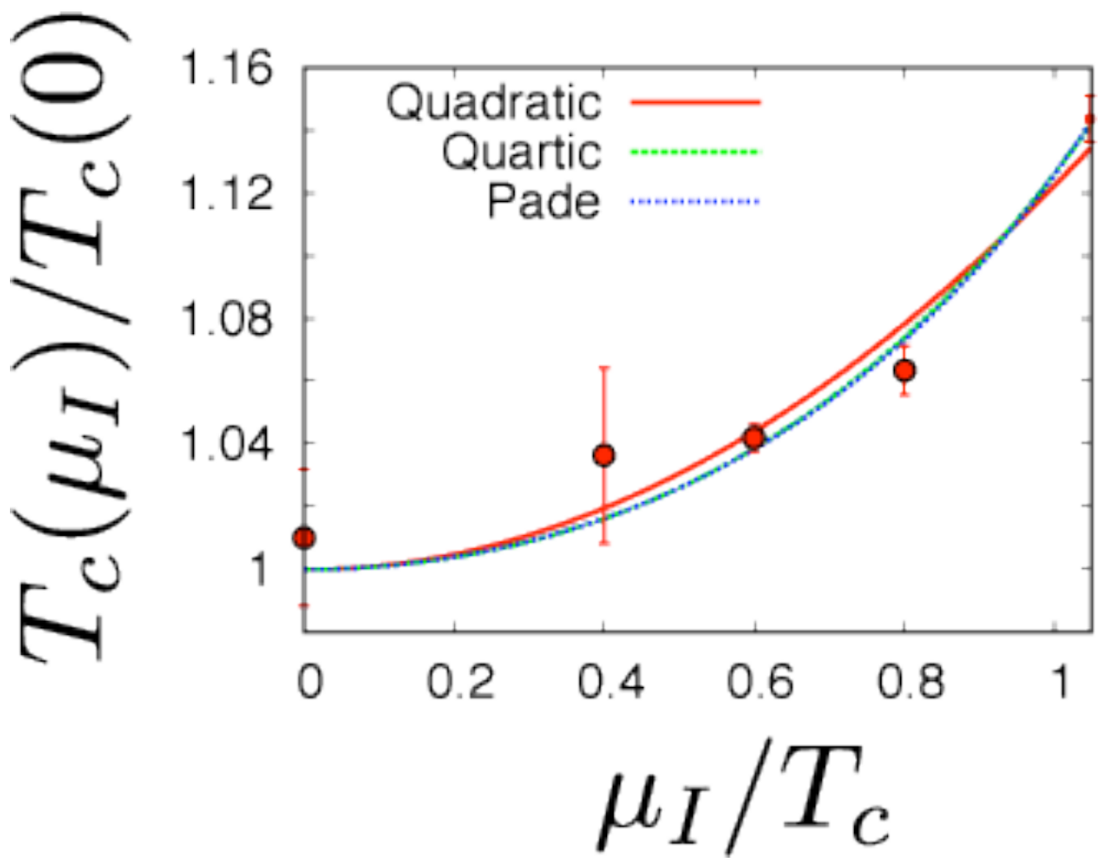
$$\frac{T_c(\mu_I)}{T_c(0)} = 1 + d_1 \left(\frac{\mu}{T_c}\right)^2$$

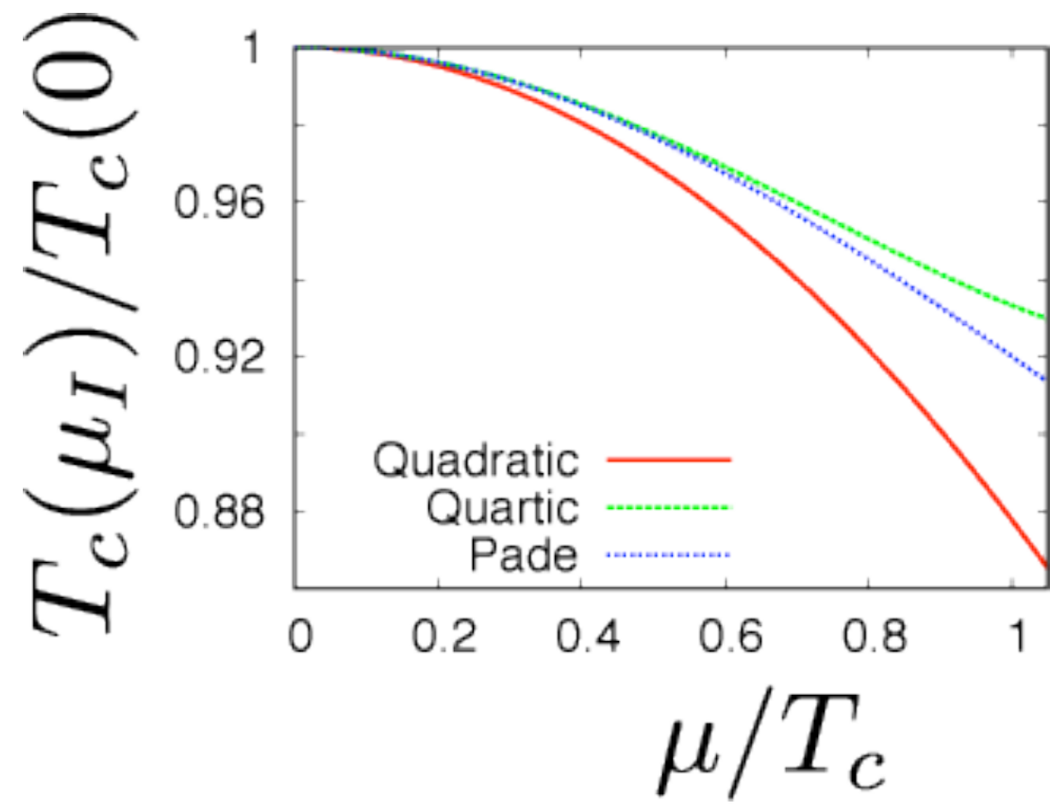
$$d_1 = 0.062 \pm 0.010$$



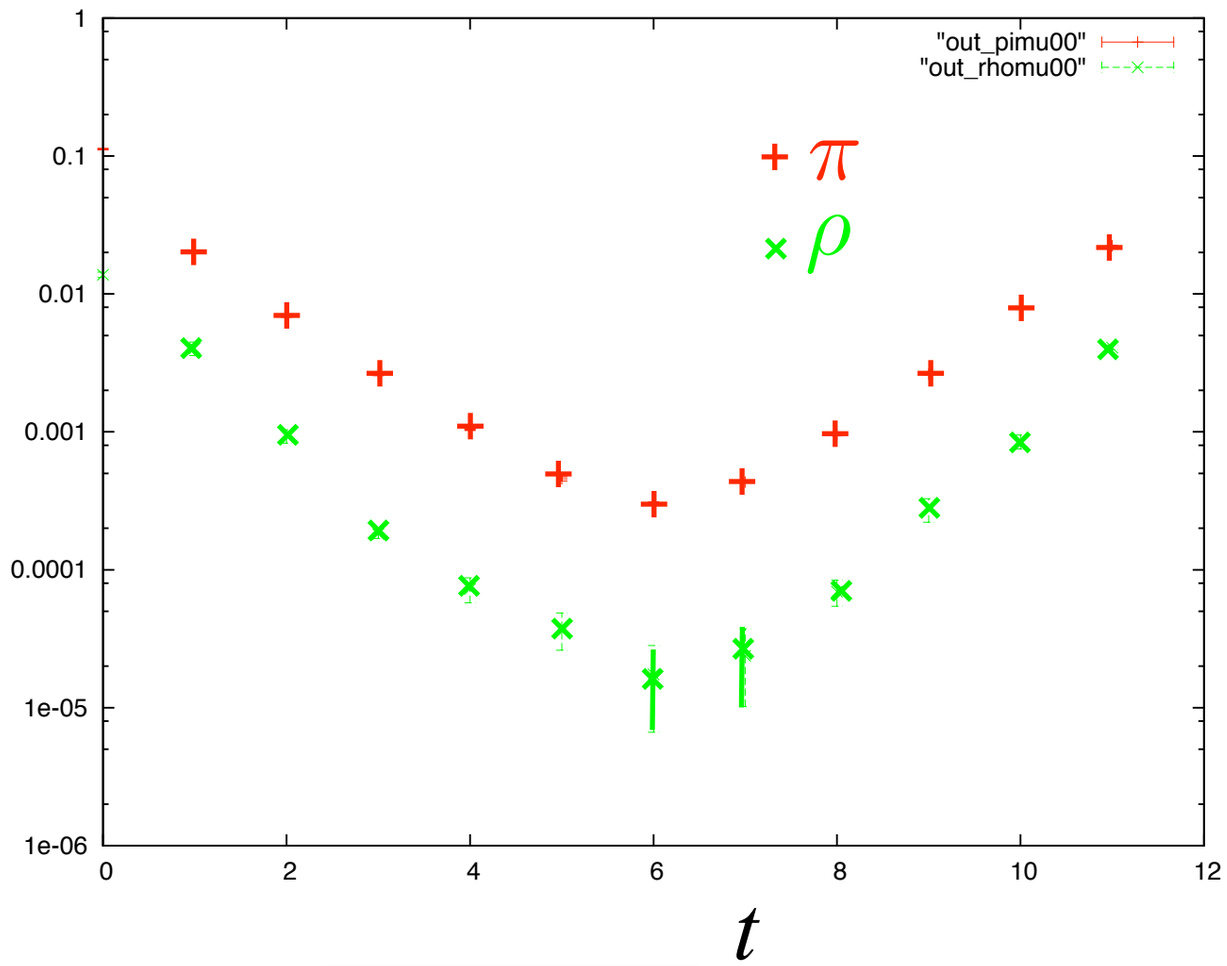


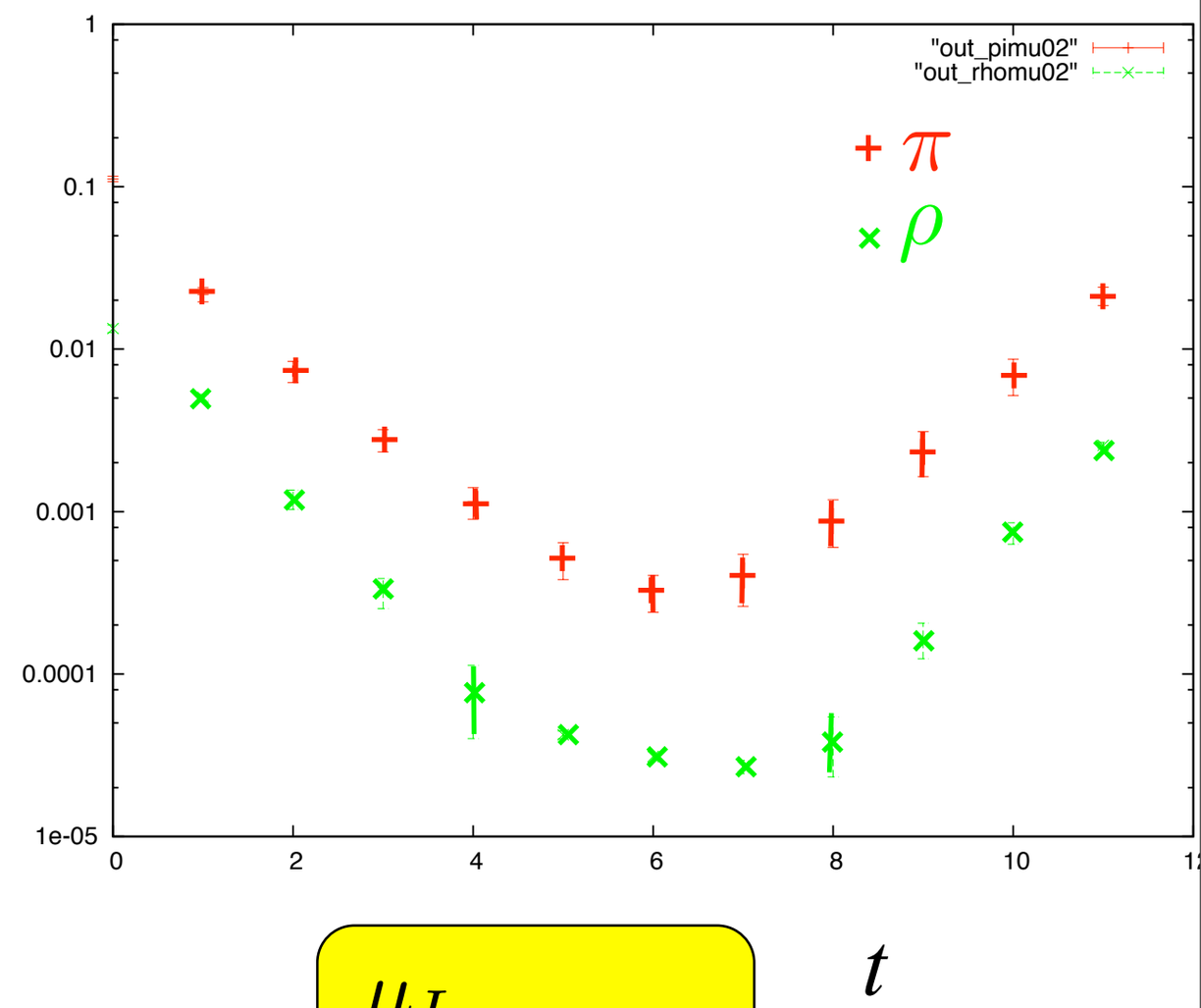
Pade $\beta_{pc} = c_0 \dfrac{1 + c_1 \mu_I^2}{1 + c_2 \mu_I^2}$


$T_c(\mu_I)/T_c(0)$
1.16
1.12
1.08
1.04
1
0
0.2
0.4
0.6
0.8
1
Quadratic
Quartic
Pade
μ_I/T_c


$T_c(\mu_I)/T_c(0)$
1
0.96
0.92
0.88
0
0.2
0.4
0.6
0.8
1
Quadratic
Quartic
Pade
μ/T_c

pi and rho propagators



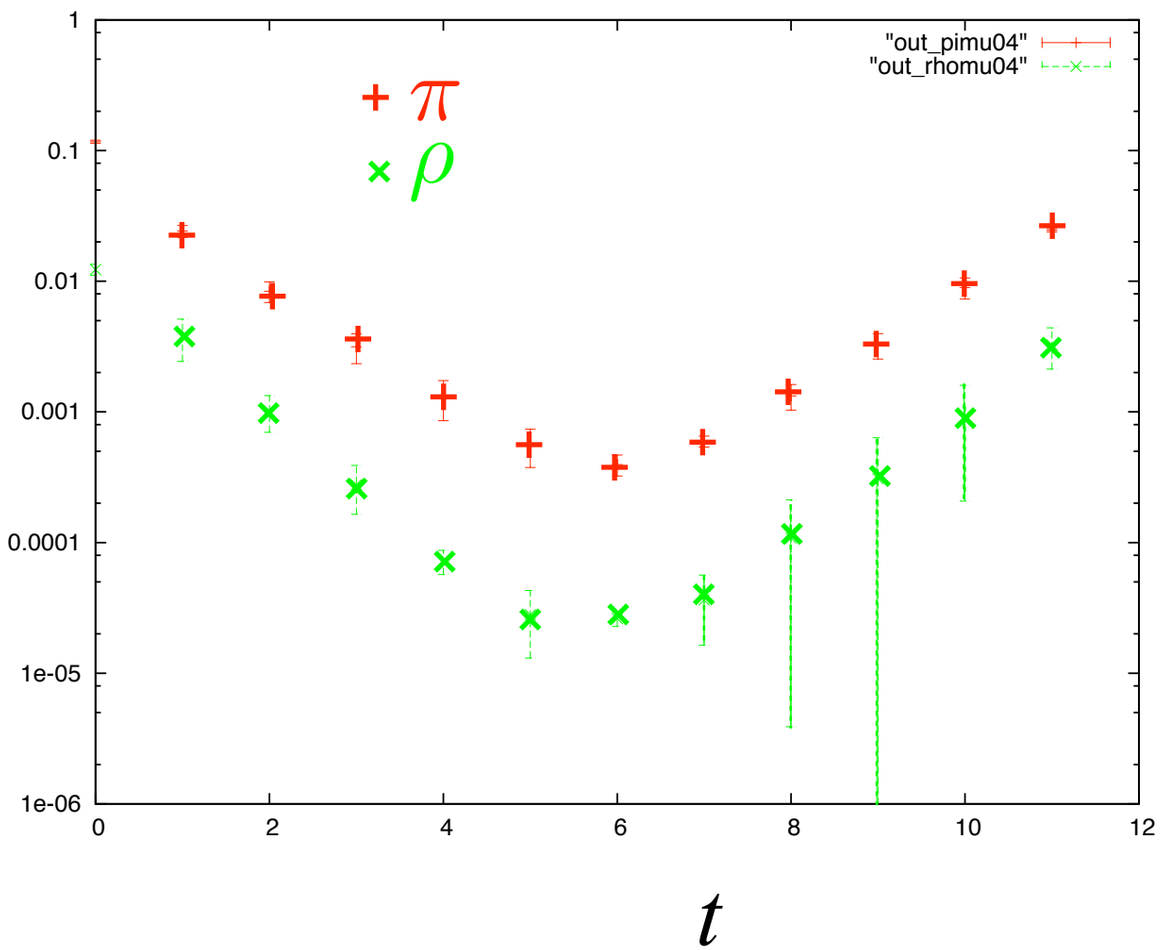


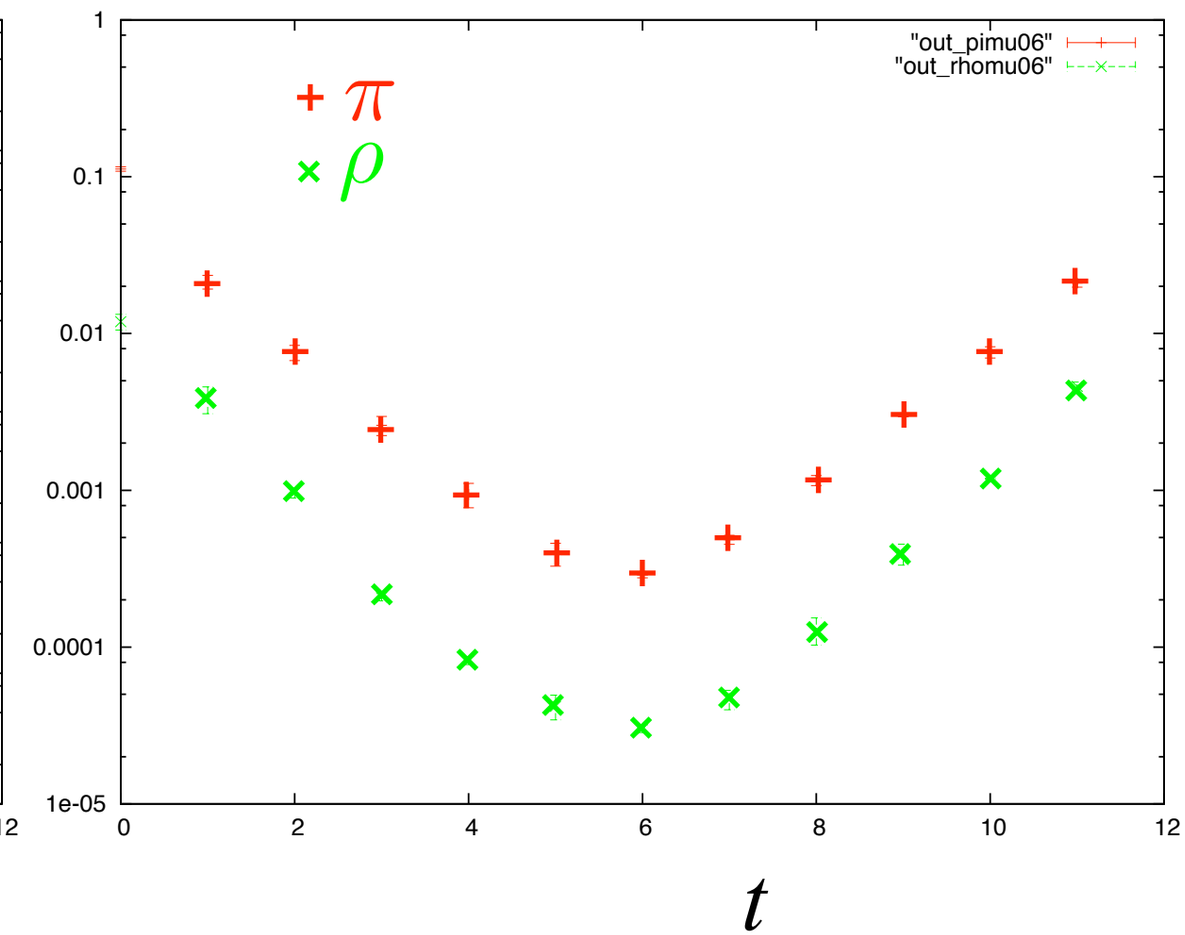
$$\frac{\mu_I}{T} = 0.0$$

$$\frac{\mu_I}{T} = 0.24$$

$$6 \times 6 \times 6 \times 12 \quad T \sim \frac{1}{3} T_c \qquad m_{PS}/m_V = 0.65 \sim 0.8$$

pi and rho propagators (2)





$$\frac{\mu_I}{T} = 0.48$$

$$\frac{\mu_I}{T} = 0.72$$

$$6 \times 6 \times 6 \times 12$$

To Do List (1)

We (Nagata and I) must learn how to find good Reweighting parameters (often only a few out of 100 configurations contribute), and how to be sure the overlap is large enough.

We must check the fluctuation of Canonical Ensamble (obtained by the Fugacity expansion) coefficients is under controle.

Trials to increase Signal/Noise ration,

☆ e.g., Fugacity expansion, then set $C_n = 0$ for n=3m+1, 3m+2.

To Do List (2) (after List (1))

- Larger Volume $12^3 \times 6$
- Smaller quark mass
- To Investigate QCD Phase using both real and imaginary chemical potential.
- Canonical Ensamble approach