5. thermodynamic quantities

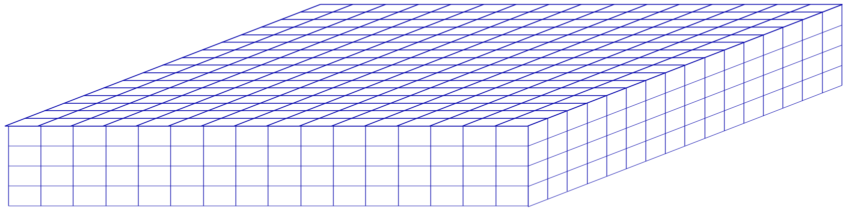
EOS

$$\epsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial T^{-1}}, \quad p = T\frac{\partial \ln Z}{\partial V} \quad \text{with} \quad Z = \mathrm{Tr} e^{-H/T} = \int_{b.c.} \mathcal{D}\phi \, e^{-S}$$

To continuously vary T and V independently, we need to (temporally) introduce anisotropic lattice.

$$a_s \neq a_t$$

$$1/T = N_t a_t, \quad V = (N_s a_s)^3$$



This requires anisotropic beta functions for the variation of a_s and a_t independently. They can in principle be estimated by exploring observables in high-dimensional anisotropic coupling parameter space through a systematic study on anisotropic lattices. --- But not easy.

To avoid anisotropic beta functions, the methods discussed in the following subsections are usually adopted.

A crucial point to be noted is that the combination

$$T^{-1}\frac{\partial}{\partial T^{-1}} + 3V\frac{\partial}{\partial V} \propto a_t\frac{\partial}{\partial a_t} + a_s\frac{\partial}{\partial a_s}$$

is nothing but a uniform scale transformation, and thus can be evaluated on just isotropic lattices too.

EOS (fixed-*Nt* approach)

$$\epsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial T^{-1}}, \quad p = T\frac{\partial \ln Z}{\partial V}$$ with $$Z = \mathrm{Tr}\, e^{-H/T} = \int_{b.c.} \mathcal{D}\phi\, e^{-S}$$

Trace anomaly

$$\frac{\epsilon - 3p}{T^4} = N_t^4 \left\{ \left\langle a\frac{dS}{da} \right\rangle - \left\langle a\frac{dS}{da} \right\rangle_{T=0} \right\} = \frac{N_t^3}{N_s^3} \sum_i a\frac{db_i}{da} \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}$$

lattice beta functions along LCP

$b = (\beta, \kappa_{ud}, \kappa_s, \cdots) \equiv (b_1, b_2, \cdots)$

measured by the simulation.
T=0 subtraction for renormal.

Integral method for *p*

Differentiate and integrate a thermodyn. relation $p = (T/V)\ln Z$

$$p = \frac{T}{V}\int_{b_0}^{b} db\, \frac{1}{Z}\frac{\partial Z}{\partial b} = -\frac{T}{V}\int_{b_0}^{b} \sum_i db_i \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}$$

such that $p(b_0) \approx 0$

numerical integration
in the coupling param. space

EOS (fixed-*Nt* approach)

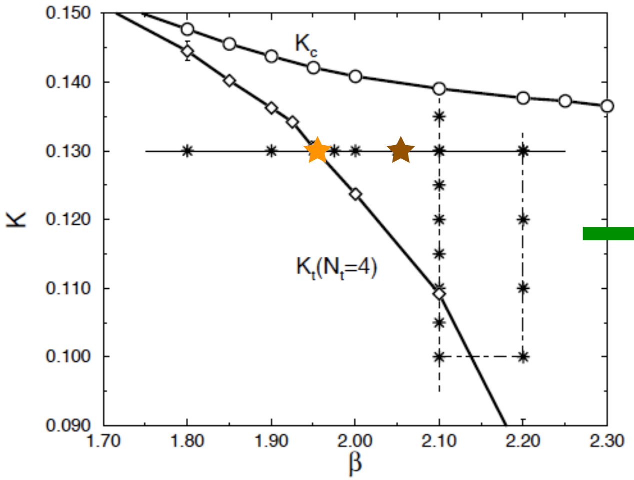
Integral method for *p*

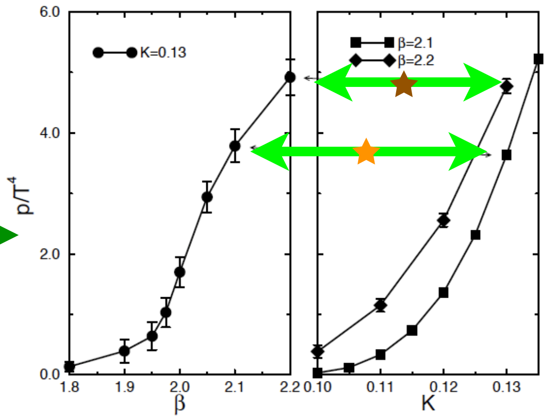
$$p = \frac{T}{V}\int_{b_0}^{b} db\,\frac{1}{Z}\frac{\partial Z}{\partial b} = -\frac{T}{V}\int_{b_0}^{b}\sum_i db_i\left\{\left\langle\frac{\partial S}{\partial b_i}\right\rangle - \left\langle\frac{\partial S}{\partial b_i}\right\rangle_{T=0}\right\}$$

such that $p(b_0) \approx 0$

numerical integration
in the coupling param. space

The integration path is free to choose as far as $p(b_0) \approx 0$





[CP-PACS: PRD64, 074510 (2001)]

- RG-improved gauge + Nf=2 clover-improved Wilson
- m_{PS}/m_V = 0.65-0.95 ($m_\pi \approx$ 600-1000 MeV)
- N_t = 4, 6

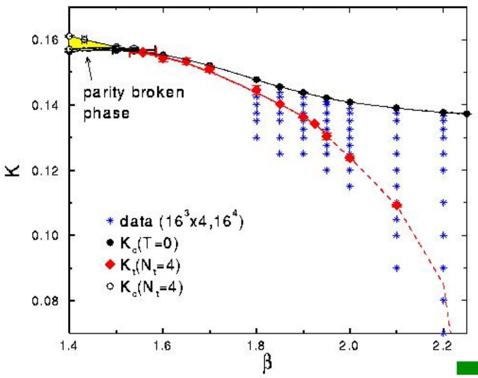
EOS (fixed-*Nt* approach)

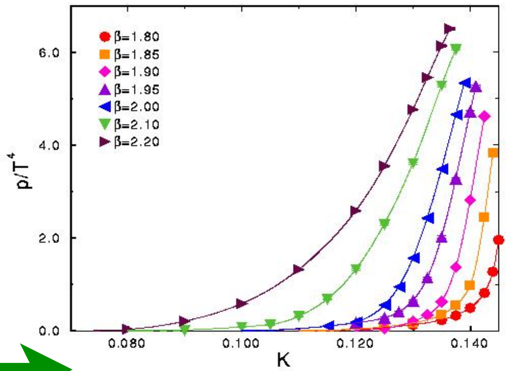
An alternative test:

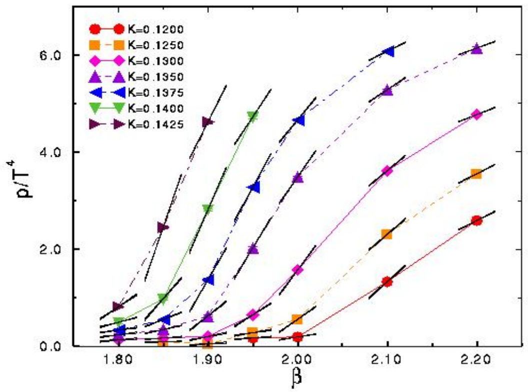
From the previous test, we learn that integration in κ leads to smaller errors.

[CP-PACS: PRD64, 074510 (2001)]

- RG-improved gauge + Nf=2 clover-improved Wilson
- m_{PS}/m_V = 0.65-0.95 ($m_\pi \approx$ 600-1000 MeV)
- N_t = 4, 6





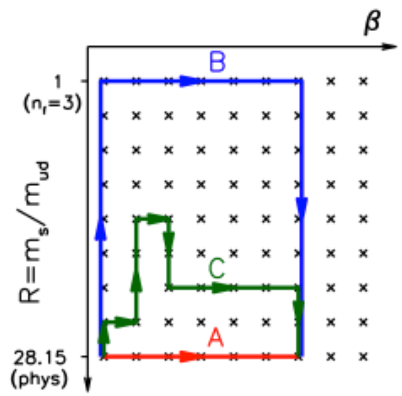


$$\frac{\partial(p/T^4)}{\partial K} = -N_t^4 \left\langle \frac{1}{N_s^3 N_t} \frac{\partial S}{\partial K} \right\rangle_{\rm sub}$$

Consistent with the slope measured directly from $\frac{\partial(p/T^4)}{\partial \beta} = -N_t^4 \left\langle \frac{1}{N_s^3 N_t} \frac{\partial S}{\partial \beta} \right\rangle_{\rm sub}$

Generalized method taking into account all possible path'

Borsanyi et al. [arXiv:1007.2580].



beta functions

$$a\frac{db_i}{da} \quad \text{with} \quad b = (\beta, \kappa_{ud}, \kappa_s, \cdots) \equiv (b_1, b_2, \cdots)$$

In the multi-dimensional parameter space of QCD, we first need to know the line of constant physics (LCP) for the world underinvestigation.

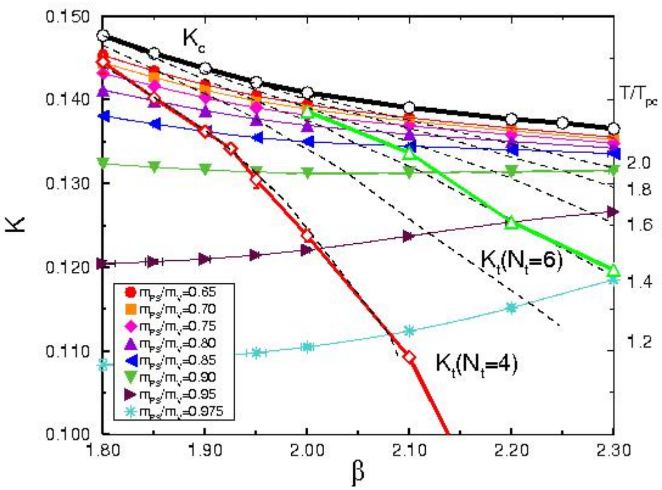
LCP

In the scaling region, LCP is defined as the points where the dimension-less ratios of observables are the same at T=0.

Different LCP's represent different world (different proton mass, different electron mass, etc.). On a LCP, different point corresponds to the same world with different lattice spacings.

Off the scaling region, precise location of LCP depends on the definition.

[example] CP-PACS, PR D64('01)074510
Iwasaki gauge + N_F=2 clover-improved Wilson quarks
LCP: $m_{PS}/m_V(T{=}0)$ = constant
Lines of constant T/T_{pc} for N_t=4
where T_{pc} determined on the same LCP



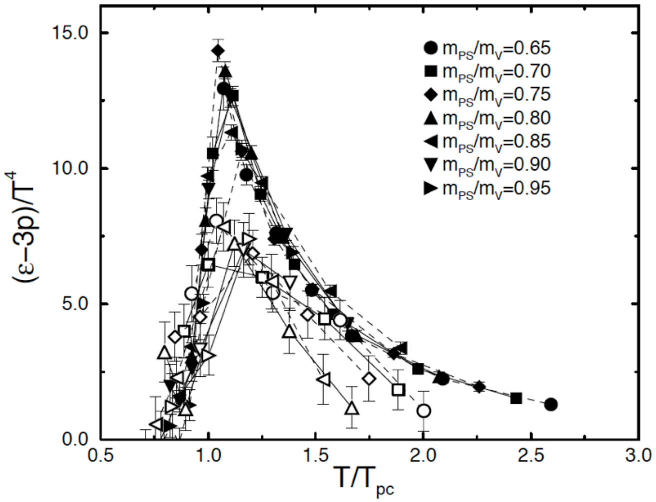
Beta functions are defined as the change of b_i along LCP.

EOS (fixed-N_t approach)

Results for N_F=2 with clover-improved Wilson quarks

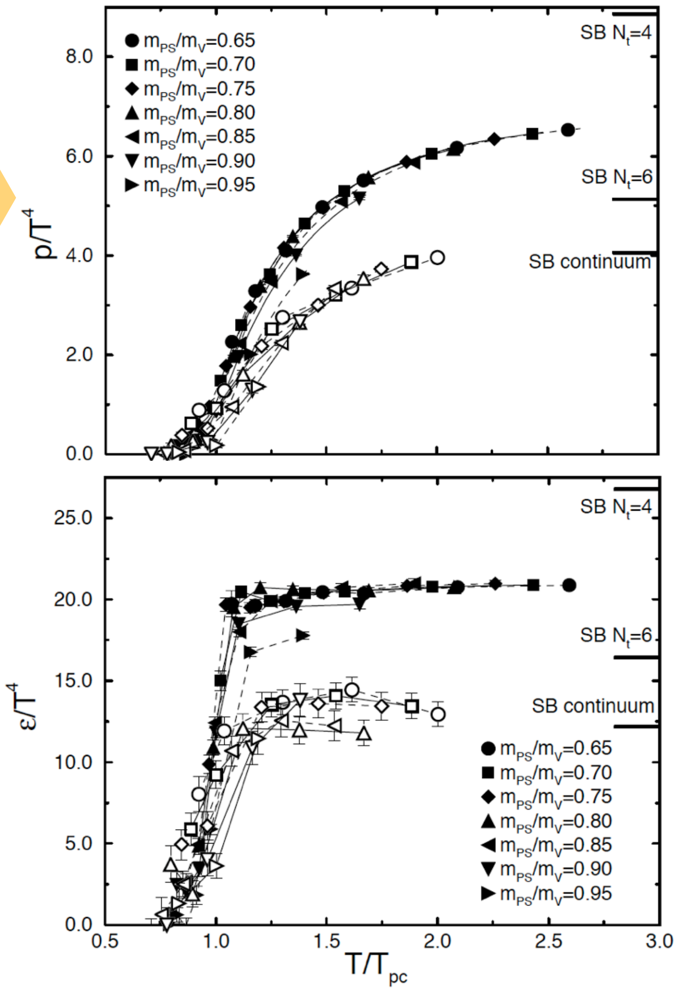
AliKhan et al. (CP-PACS) PRD64('01)

- RG-improved gauge + clover-improved Wilson
- m_{PS}/m_V = 0.65-0.95 ($m_\pi \approx$ 600-1000 MeV)
- N_t = 4, 6



$$p = \frac{T}{V}\int_{b_0}^{b} db \frac{1}{Z}\frac{\partial Z}{\partial b}$$

integral method

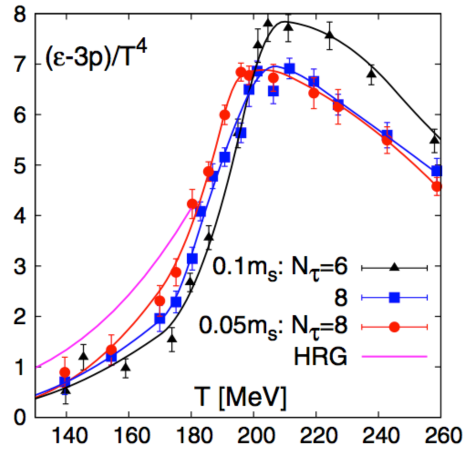


EOS (fixed-*Nt* approach)

Recent results for N_F=2+1 with various improved staggered quarks

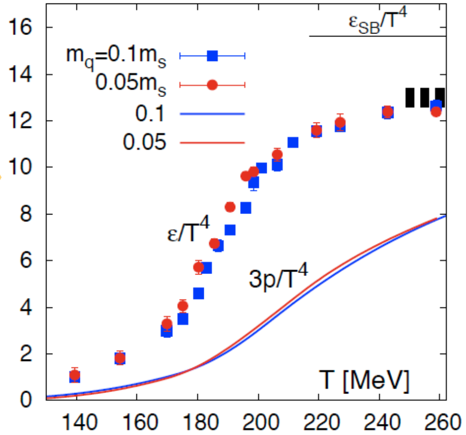
Chen et al. (HotQCD) PRD81('10)

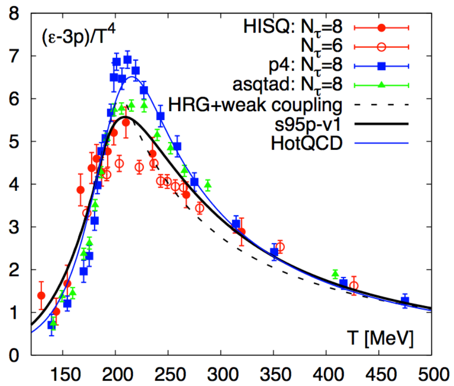
- tree-level Symanzik gauge + p4-improved stag.
- $m_s \approx$ "physical", $m_l/m_s = 0.05$ ($m_\pi^{pNG} \approx 154$ MeV)
- $N_t = 8$

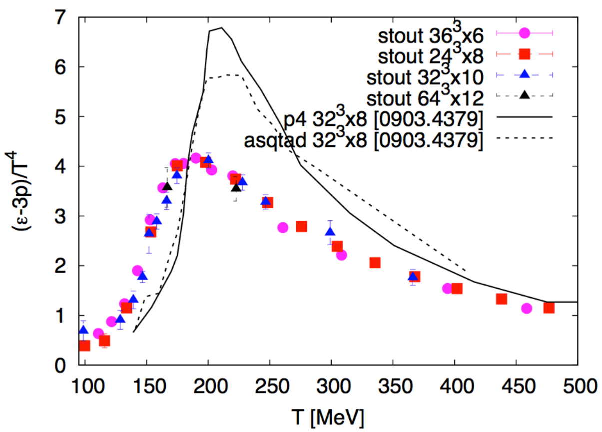


$$p = \frac{T}{V}\int_{b_0}^{b} db \frac{1}{Z}\frac{\partial Z}{\partial b}$$

integral method







Difference among different stag. quarks = errors due to the remaining taste violation

HotQCD and Wuppertal-Budapest proceedings of Lattice 2010

EOS (fixed-scale approach)

(Our) motivations

- Results from staggered-type quarks should be cross-checked by other lattice quarks whose theoretical basis is rigid.
- Conventional EOS calculation requires a large scale systematic simulation, and is still expensive with Wilson-type and chiral lattice quarks.

A large fraction of the cost <= **T = 0 simulations**

- Determination of basic information about the simulation point: (lattice scale, etc.)
- Determination of LCP, and non-perturbative beta functions at all **$T > 0$** simulation points
- **$T = 0$** subtractions => **$T = 0$** simulations needed at all **$T > 0$** simulation points

With the fixed scale approach, all $T > 0$ simulations are done at the same point of the coupling parameter space.

- ☑ All the simulations are automatically on the same LCP.
- ☑ **$T = 0$** simulations needed only at one point.
- ☑ Scale, non-perturbative beta functions, etc. needed only at this point too.

We may reduce the cost for $T = 0$ simulations.

EOS (fixed-scale approach)

$$\epsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial T^{-1}}, \quad p = T\frac{\partial \ln Z}{\partial V} \quad \text{with} \quad Z = \mathrm{Tr} e^{-H/T} = \int_{b.c.} \mathcal{D}\phi \, e^{-S}$$

Trace anomaly

$$\frac{\epsilon - 3p}{T^4} = N_t^4 \left\{ \left\langle a\frac{dS}{da} \right\rangle - \left\langle a\frac{dS}{da} \right\rangle_{T=0} \right\} = \frac{N_t^3}{N_s^3} \sum_i a\frac{db_i}{da} \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}$$

lattice beta functions along LCP

$b = (\beta, \kappa_{ud}, \kappa_s, \cdots) \equiv (b_1, b_2, \cdots)$

measured on the lattice
T=0 subtractions for renorm.

Because all the simulations are done at one point in the coupling parameter space, the conventional integral method in the coupling parameter space is not applicable.

EOS (fixed-scale approach)

T-integration method for *p*

Umeda et al., PRD79, 051501 ('09)

Using a thermodyn. relation at μ=0:

$$T\frac{\partial}{\partial T}\left(\frac{p}{T^4}\right) = \frac{\epsilon - 3p}{T^4} \quad \Rightarrow \quad \frac{p}{T^4} = \int_{T_0}^{T} dT \, \frac{\epsilon - 3p}{T^5}$$

$p(T_0) \approx 0$

Disadvantages/challenges:

- The resolution in *T* is limited due to the discreteness of *Nt*.

  => interpolation in *T*

  <= *a* should be sufficiently small / odd *Nt* programs /

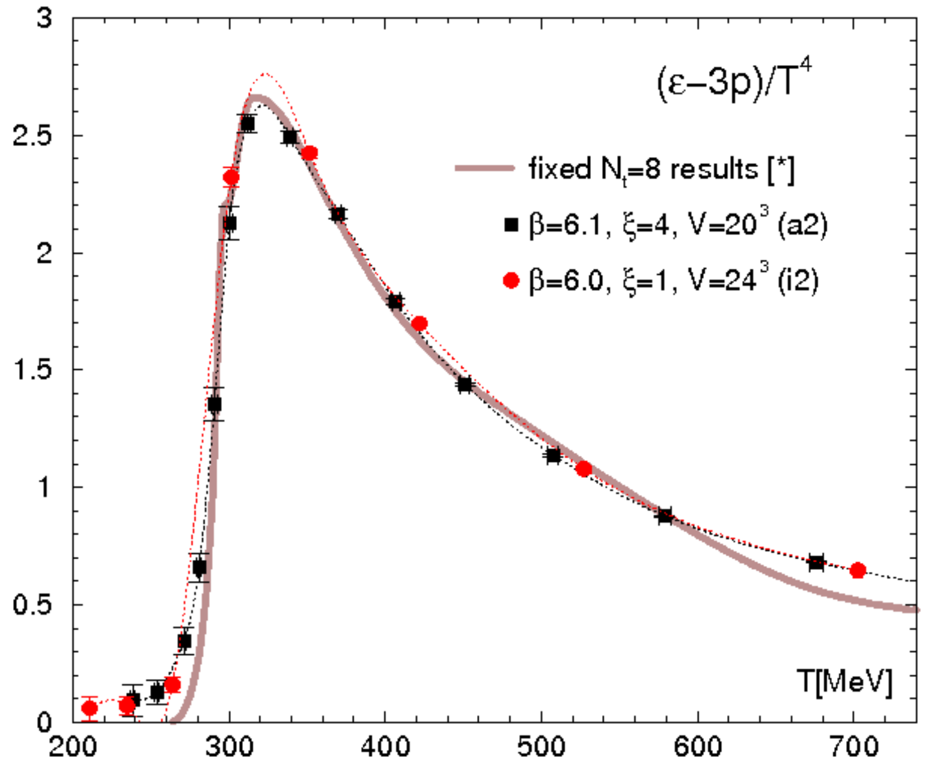
  combine different ß in a scaling region / ...

- Large statistics required at low *T* (large *Nt*) to compete a big cancellation by the *T*=0 subtraction

EOS (fixed-scale approach)

Test in quenched QCD

Umeda et al., PRD79, 051501 ('09)



Results compared for

★ isotropic lattices
(*as*~0.095fm, *Nt*=3-10 =>
T=200-700MeV, *Ls*~1.5fm)

★ anisotropic lattice with ξ=4
(4-times smaller *at* => 4-times finer *T*-resolution)

★ result of the fixed *Nt* approach
(*Nt*=8 by Boyd el al. NPB469(96):
Ns=32 => *Ls*~2.7fm around *Tc*)

Note: effects due to small *Ls* are physical finite volume effects, i.e not a matter of the algorithm.

Besides understandable deviations, results consistent with each other

➡ consistent with the fixed *Nt* approach

➡ *T*-interpolation under control on the isotropic lattice

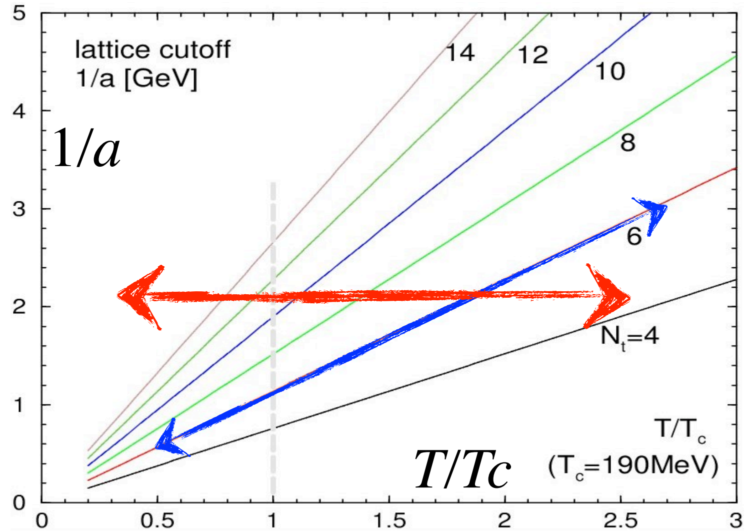
➡ computation costs much reduced

fixed scale approach vs. fixed *Nt* approach

Pros and cons:

- A common T=0 simulation enough for all T=0 subtractions.
  - We can even borrow publicly available high statistic configurations on ILDG
- Automatically on a LCP w/o fine tuning.

→ T=0 simulation costs redusable.



- The resolution in T is limited because *Nt* is discrete. => under control for EOS (see Umeda et al., PRD79, 051501 ('09))

At high T: (T>2-3Tc)

- *Nt* too small => another source of errors
- Keep the lattice volume large.

Low T / around Tc :

- More costs due to larger *Nt*.
- Keep the lattice spacing small.

complementary! ⟷

fixed *Nt* approach

powerful at high T
coarse at low T

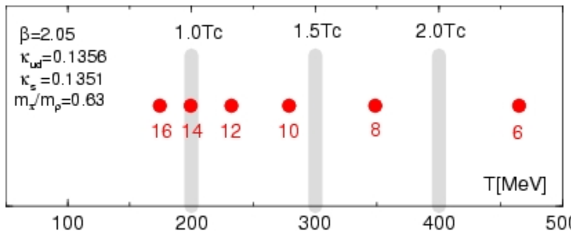
EOS (fixed-scale approach)

Trial calculation in N_F=2+1 QCD

Umeda et al. (WHOT-QCD Collab.)

- T=0 simulation: on $28^3 \times 56$ by CP-PACS/JLQCD Phys. Rev. D78 (2008) 011502
  - RG-improved Iwasaki glue + clover-improved Wilson quarks
  - β=2.05, κ_{ud}=0.1356, κ_s=0.1351 ($m_\pi \sim 634\mathrm{MeV}, \ \frac{m_\pi}{m_\rho} = 0.63, \ \frac{m_{\eta_s s}}{m_\phi} = 0.74$)
  - V~(2 fm)3 , a=0.07 fm,
  - configurations available on the ILDG/JLDG
- T>0 simulations: on $32^3 \times Nt$ (Nt=4, 6, ..., 14, 16) lattices

  RHMC algorithm, same parameters as the T=0 simulation

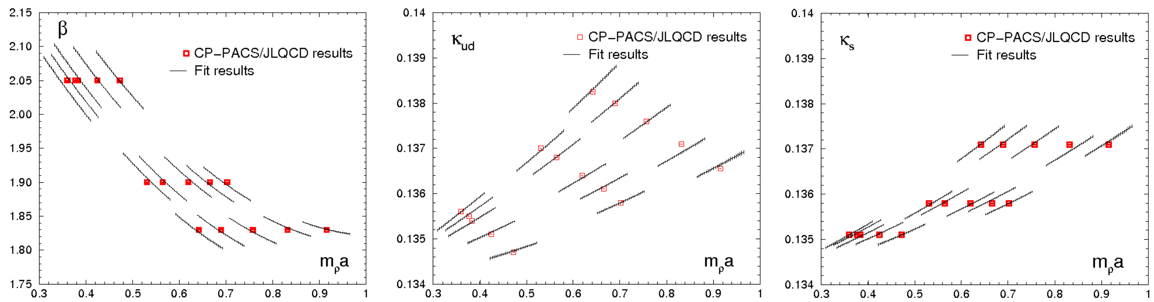


Beta functions <= T=0 meson mass data at 3 (β) × 5 (κ_{ud}) × 2 (κ_s) = 30 data points

fit $\beta, \kappa_{ud}, \kappa_s$ as functions of $(am_\rho), \left(\frac{m_\pi}{m_\rho}\right), \left(\frac{m_{\eta ss}}{m_\phi}\right)$ => $a\frac{db}{da} = am_\rho \left.\frac{\partial b}{\partial(am_\rho)}\right|_{\mathrm{LCP}}$

with LCP defined by

$$\frac{m_\pi}{m_\rho} = \mathrm{const.}, \quad \frac{m_{\eta ss}}{m_\phi} = \mathrm{const.}$$

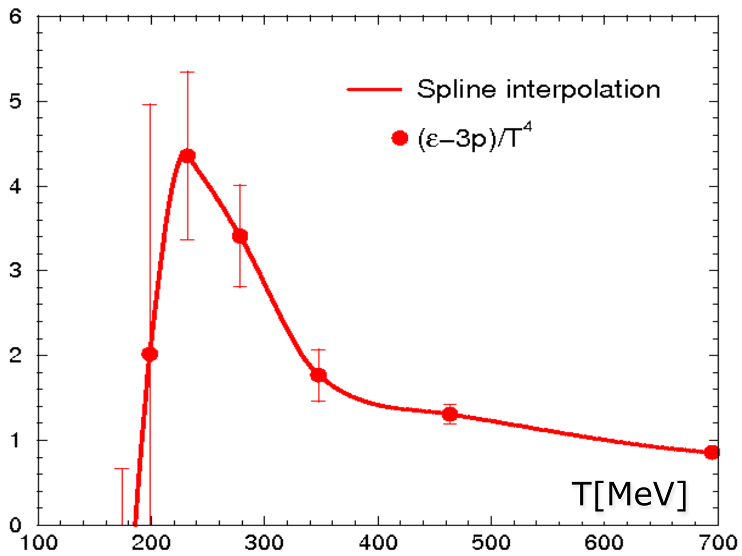


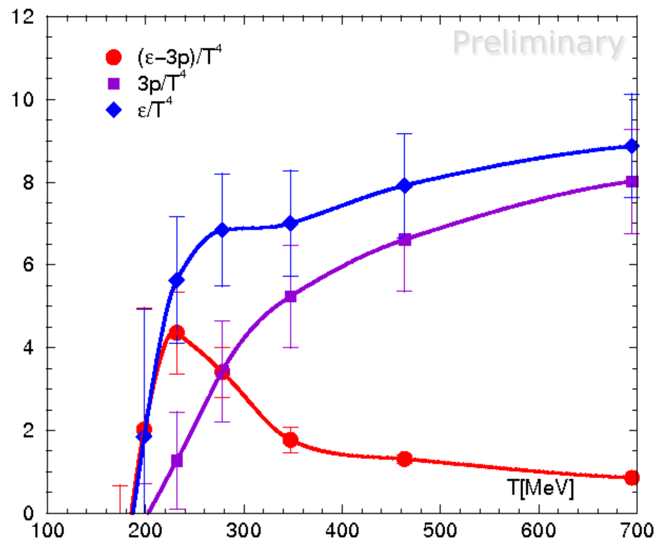
$\left(a\frac{\partial\beta}{\partial a}, \ a\frac{\partial\kappa_{ud}}{\partial a}, \ a\frac{\partial\kappa_s}{\partial a}\right)_{\text{simulation point}} = (-0.334(4), \ 0.00289(6), \ 0.00203(5))$ (statistical errors only)

EOS (fixed-scale approach)

Trial calculation in N_F=2+1 QCD

EOS (preliminary) Umeda et al. (WHOT-QCD Collab.) @ Lattice 2010



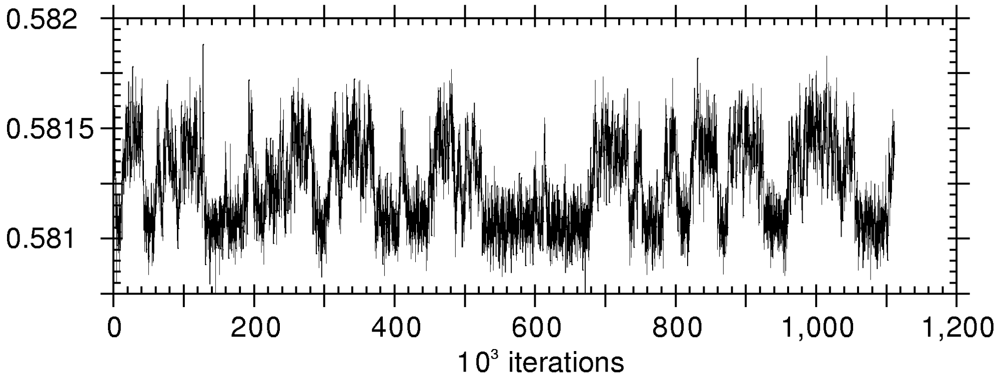


More works needed

- more statistics at low *T*
- better *T*-resolution => odd *Nt* ? / combine with other ß on the LCP
- more refined method to evaluate beta functions? => reweighting?
- just on the physical point (using T=0 configurations generated by the PACS-CS Collab.)

Latent heat

When the transition is 1st order,



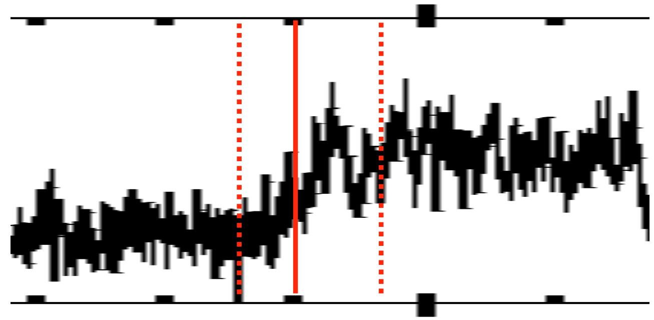
Plaquette history at *Tc*

Iwasaki et al., PR D46('92)4657

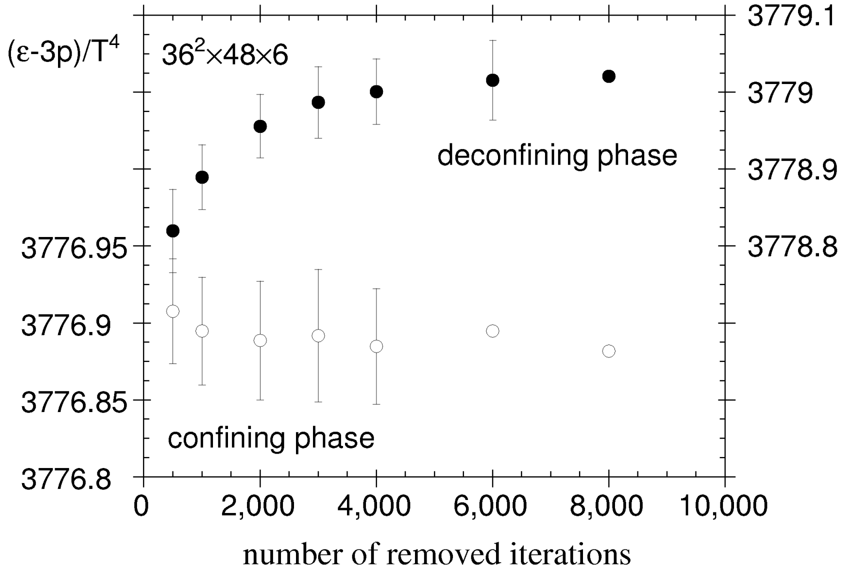
36^2x48x6, 1150.000 iterations

bin=100

Flip-flop among two phases visible.



remove transition states



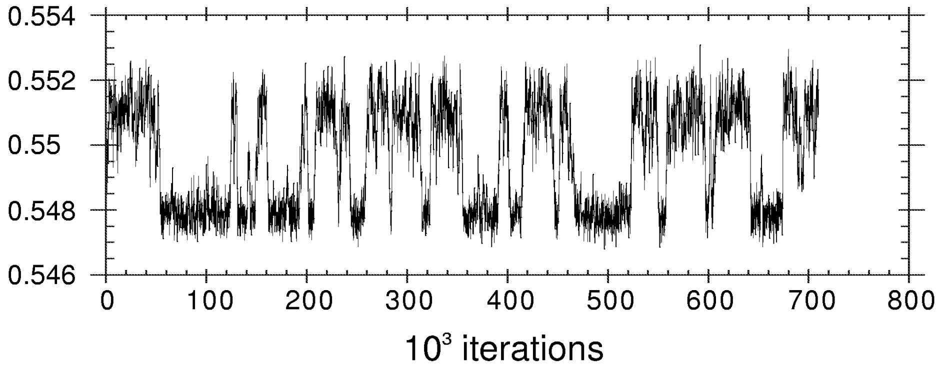
p is continuous at *Tc* => $\Delta\varepsilon$

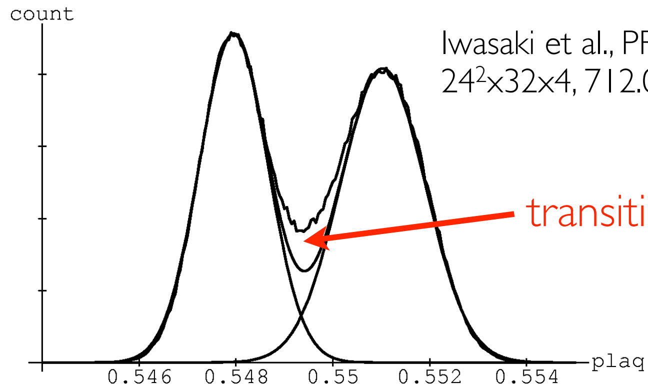
results for SU(3) YM theory:

	Nt	$\Delta\varepsilon/T_C^4$
Standard	4	2.27(5)
	6	1.53(4)
Symanzik	4	1.40--1.57 (9--12)

Interface tension

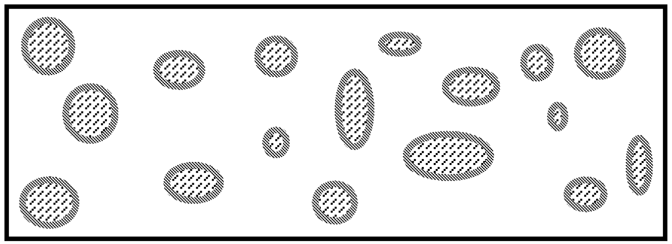
When the transition is 1st order,



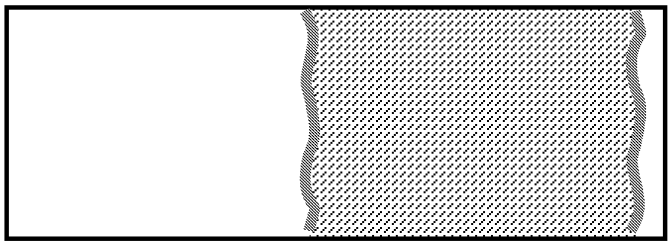


We now extract information from the transition states.

When two phases coexists with a non-zero interface tension:

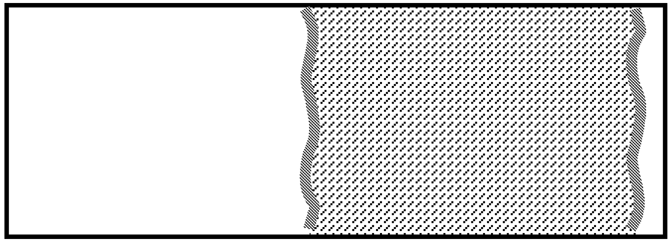


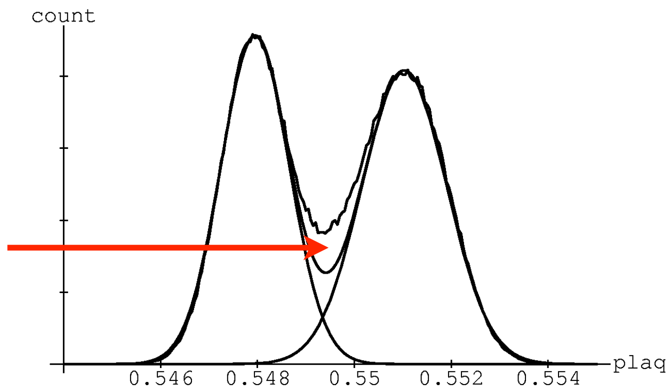
Large interface area
=> less probable



Minimum interface area with p.b.c
=> dominant contribution

Interface tension





Interface area approximately known.
=> The probability to have such transition state $\propto e^{-\sigma_I A}$

In actual calculation, we take into account the effects of

* lattice geometry
* parallel transports
* capillary wave collections on the interface

See Iwasaki et al., PR D49('94)3540 for details.

For the case of SU(3) YM, $\sigma_I / T_C^{\,3} \approx 0.15 - 0.16$

Many other interesting quantities

entropy density / sound velocity / ...

transport coefficients / ...

heavy quark potential / screening masses / effective couplings / ...

spectral functions / ...

to investigate
dissociation of charmonia / effective masses and decay rates of them at T>0 / ...

Young powers and many new ideas are starved for!!

thank you

What will happen at $\mu \neq 0$?

Similar to the high *T* case,

- restoration of the chiral symmetry

  when the thermal energy > potential barrier between the sectors.

  $\mu \sim M_N / 3$

- breakdown of confinement

  high density => short average distances between quarks

  asymptotic freedom: $g(\mu) \longrightarrow 0$ *as* $\mu \longrightarrow \infty$.

  $\mu \sim \Lambda_{QCD} \sim O(100)$ MeV

=> visit the lectures by Atsushi Nakamura