Beyond Deterministic Fits: A Bayesian Exploration of Bc Spectroscopy

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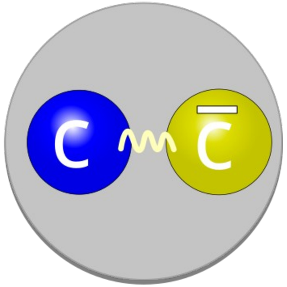
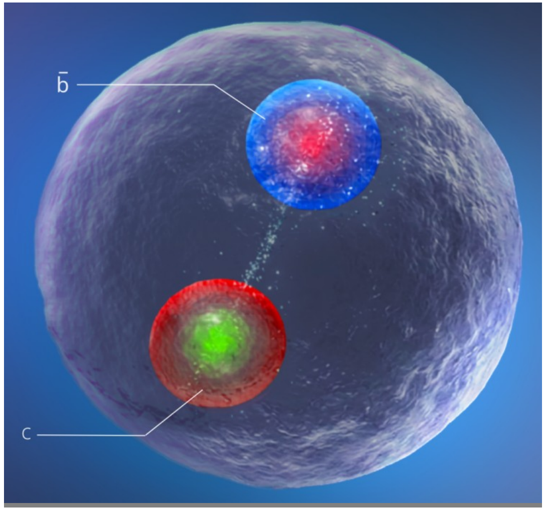
Outline

- Introduction
- Deterministic Fits
- Bayesian Parameter Estimation
- B_c Spectroscopy and Confinement
- Closing thoughts

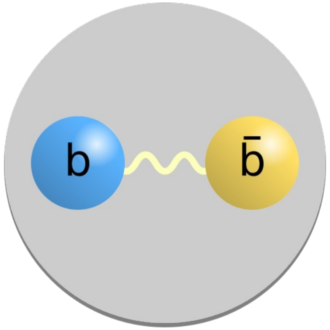
Based on
C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Why B_c?





- Unique heavy-heavy system
- Unequal masses
- Rich spectroscopy
- Growing experimental information
  - Orbitally excited states (Phys. Rev. Lett. 135 (2025) 231902, arXiv:2507.02149 [hep-ex])
  - B_c* (arXiv:2605.16228 [hep-ex])
- Confinement?



The usual spectroscopy workflow

Data → **Fit parameters** → **One parameter set** → **Predictions**

B_c less data.

Deterministic.

Correlations?

Uncertainties?

Potential I

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma r + V_c$$

CORNELL POTENTIAL

SRWE

$$u''_{nl}(r) + 2\mu\left[E_{nl} - V(r) - \frac{l(l+1)}{2\mu r^2}\right]u_{nl}(r) = 0$$

$$M = m_1 + m_2 + E_{nl}$$

$$V_{SS}(r) = \frac{32\pi\alpha_s}{9m_1m_2}\mathbf{S}_1\cdot\mathbf{S}_2\,\delta(r)$$

$$V_{LS}^{(+)\mathrm{I}}(r) = \frac{1}{4m_1^2m_2^2r}\left[\left((m_1+m_2)^2 + 2m_1m_2\right)\frac{4\alpha_s}{3r^2} - (m_1^2+m_2^2)\sigma\right]\mathbf{L}\cdot\mathbf{S}_+$$

$$V_T(r) = \frac{\alpha_s}{3m_1m_2r^3}S_{12}$$

Spin-dependent Interactions

$$V_{LS}^{(-)\mathrm{I}}(r) = \frac{(m_2^2-m_1^2)}{4m_1^2m_2^2r}\left[\frac{4\alpha_s}{3r^2} - \sigma\right]\mathbf{L}\cdot\mathbf{S}_-$$

State Mixing

Smeared Gaussian

$$\delta(r) \to \left(\frac{\rho}{\sqrt{\pi}}\right)^3 e^{-\rho^2r^2}$$

$$M(n^{2S+1}L_J)$$

$$\langle\mathbf{L}\cdot\mathbf{S}_+\rangle = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

$$\langle\mathbf{L}\cdot\mathbf{S}_-\rangle = \sqrt{\frac{(2l+3)(2l-1)}{10}}\,\delta_{J,L}$$

$$\langle\mathbf{S}_1\cdot\mathbf{S}_2\rangle = \frac{1}{2}s(s+1) - \frac{3}{4}$$

$$\langle S_{12}\rangle = \frac{4}{(2l+3)(2l-1)}\left[s(s+1)l(l+1) - \frac{3}{2}\langle\mathbf{L}\cdot\mathbf{S}\rangle - 3\langle\mathbf{L}\cdot\mathbf{S}\rangle^2\right]$$

Eichten E et al 1975 PRL 34 369–72
Eichten E et al 1976 PRL 36 1276
Eichten E et al 1980 PRD 21 203

Eichten E & Feinberg F L 1979 PRL 43 1205
Eichten E & Feinberg F 1981 PRD 23 2724
Gromes D 1984 ZPC 26 401

Lucha W & Schoberl F F arXiv:hep-ph/9601263
Barnes T et al 2005 PRD 72 054026
Ikhdair S M & Sever R 2005 IJMPA 20 4035-4054

Potential II

CORNELL POTENTIAL

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma r + V_c$$

MODIFIED CORNELL POTENTIAL

$$V_{\text{Ext}}(r) = V(r) + C_0 \ln(1 + \sigma' r)$$

$$V_{LS}^{(+)\text{II}}(r) = \frac{1}{4m_1^2 m_2^2 r}\left[\left((m_1+m_2)^2 + 2m_1 m_2\right)\frac{4\alpha_s}{3r^2} - (m_1^2 + m_2^2)\left(\sigma + \frac{C_0\sigma'}{1+\sigma' r}\right)\right]\mathbf{L}\cdot\mathbf{S}_+$$

$$V_{LS}^{(-)\text{II}}(r) = \frac{(m_2^2 - m_1^2)}{4m_1^2 m_2^2 r}\left[\frac{4\alpha_s}{3r^2} - \sigma - \frac{C_0\sigma'}{1+\sigma' r}\right]\mathbf{L}\cdot\mathbf{S}_-$$

$$M(n^{2S+1}L_J)$$

$$|nL_L\rangle' = \cos\theta_{nL}\,|n^1L_L\rangle + \sin\theta_{nL}\,|n^3L_L\rangle,$$

$$|nL_L\rangle = -\sin\theta_{nL}\,|n^1L_L\rangle + \cos\theta_{nL}\,|n^3L_L\rangle,$$

S. Godfrey and R. Kokoski, PRD 43, 1679-1687 (1991)

C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Deterministic Fits

Parameters	Set-I[a]	Set-II[b]
α_s	0.371	0.479
σ (in GeV^2)	0.184	0.149
ρ (in GeV)	3.999	1.409
V_c (in GeV)	-0.068	0.001

[a] Inputs: $B_c(1^1S_0)$, $B_c(2^1S_0)$, $\eta_b(1^1S_0)$, $\Upsilon(1^3S_1)$, $\eta_b(2^1S_0)$, $\Upsilon(2^3S_1)$, $\Upsilon(3^3S_1)$, $\Upsilon(4^3S_1)$, $\Upsilon(5^3S_1)$, $\Upsilon(6^3S_1)$, $\Upsilon(1^3S_1) - \eta_b(1^1S_0)$, $\Upsilon(2^3S_1) - \eta_b(2^1S_0)$, $h_b(1^1P_1)$, $\chi_{b0}(1^3P_0)$, $\chi_{b1}(1^3P_1)$, $\chi_{b2}(1^3P_2)$, $h_b(2^1P_1)$, $\chi_{b0}(2^3P_0)$, $\chi_{b1}(2^3P_1)$, $\chi_{b2}(2^3P_2)$, $\chi_{b1}(3^3P_1)$, $\chi_{b2}(3^3P_2)$, $\Upsilon_2(1^3D_2)$ (Navas *et al.*, 2024), and $B_c^*(1^3S_1) - B_c(1^1S_0)$ (Mathur *et al.*, 2018).

[b] Inputs: $B_c(1^1S_0)$, $B_c(2^1S_0)$, $B_c(2^1S_0) - B_c(1^1S_0)$ (Navas *et al.*, 2024), and $B_c^*(1^3S_1) - B_c(1^1S_0)$ (Mathur *et al.*, 2018).

CORNELL POTENTIAL

State	Set-I	Set-II	PDG (Navas *et al.*, 2024)
1^1S_0	6377	6274	6274.47(0.32)
1^3S_1	6445	6329	
2^1S_0	6964	6871	6871.2(1.0)
2^3S_1	7004	6892	
3^1S_0	7368	7246	
3^3S_1	7400	7261	
4^1S_0	7706	7554	
4^3S_1	7735	7566	
5^1S_0	8008	7826	
5^3S_1	8035	7836	
6^1S_0	8285	8074	
6^3S_1	8310	8083	

$$\chi^2 = \sum_k \frac{\left(M_k^{\mathrm{th}} - M_k^{\mathrm{exp}}\right)^2}{\left(\Delta M_k^{\mathrm{exp}}\right)^2}$$

$$\chi^2_{\mathrm{Set-I}} \approx 1.3 \times 10^5$$

However...

$$\chi^2_{\mathrm{Set-II}} \approx 3 \times 10^{-9}$$

Bayesian parameter estimation

Priors → **Likelihood** → **MCMC** → **Posterior**

$$\log p(\theta) = \begin{cases} 0, & \theta_{\min} \le \theta \le \theta_{\max}, \\ -\infty, & \text{otherwise}, \end{cases}$$

$$N_{\text{walkers}} = 8 \times N_{\text{parameters}}$$

$$\chi^2(\Theta) = \sum_i \left(\frac{X_i^{\text{theo}}(\Theta) - X_i^{\text{expt}}}{\Delta X_i^{\text{expt}}} \right)^2$$

$$\log \mathcal{L}(\Theta) = -\tfrac{1}{2}\chi^2(\Theta)$$

Correlations!

Uncertainties!

$$\alpha_s \in [0.20, 0.56], \qquad \sigma \in [0.10, 0.23]\ \text{GeV}^2,$$
$$V_c \in [-0.250, 0]\ \text{GeV}, \qquad \rho \in [1.00, 4.00]\ \text{GeV},$$
$$\sigma' \in [0.05, 0.50]\ \text{GeV}, \qquad C_0 \in [0.05, 0.50]\ \text{GeV}.$$

$$\log P(\Theta) = \log p(\Theta) + \log \mathcal{L}(\Theta)$$

$$m_c = 1.5\ \text{GeV}, \qquad m_b = 4.8\ \text{GeV}.$$

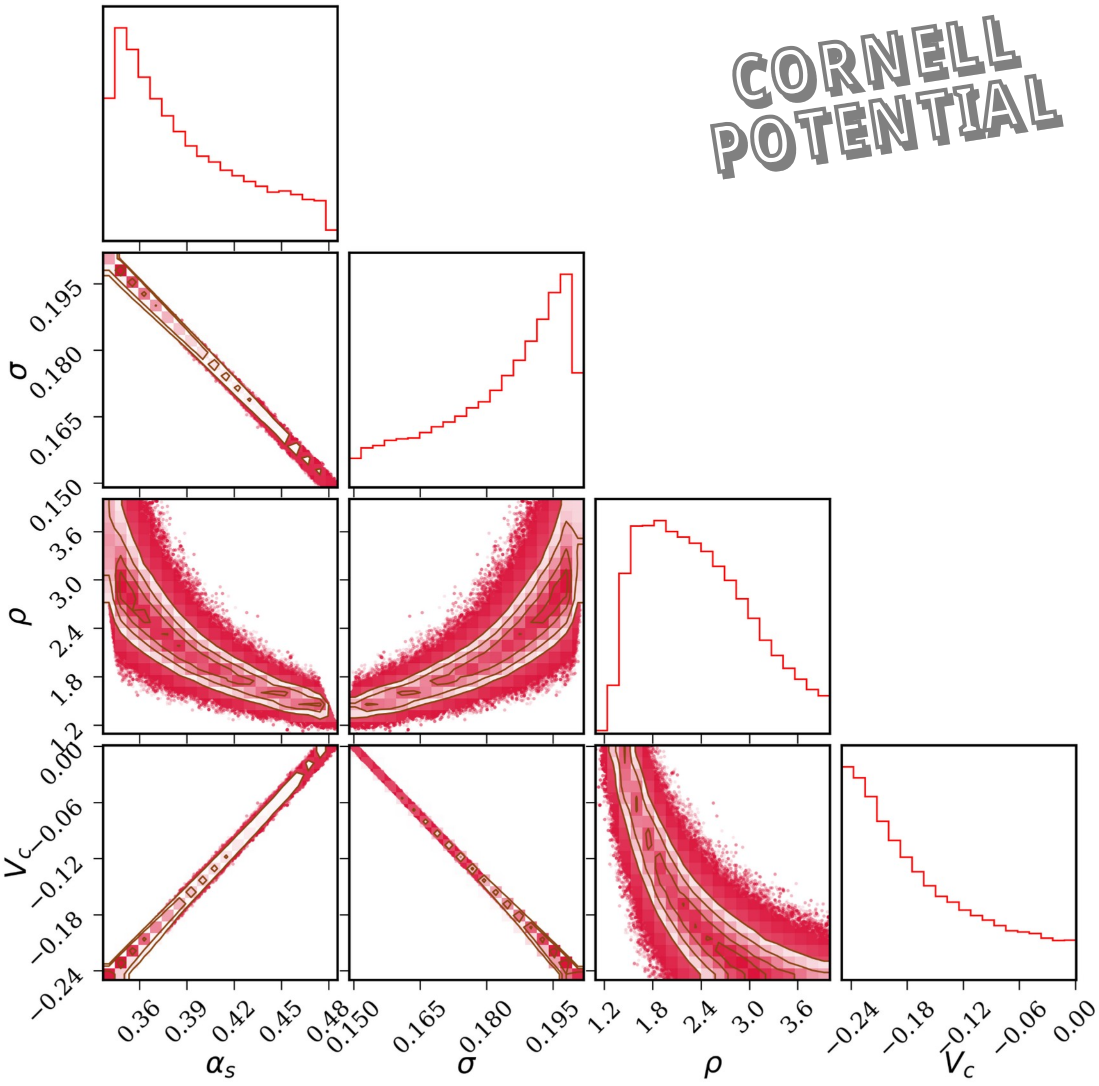
emcee package
D. Foreman-Mackey et al., PASP 125, 306-312 (2013) [arXiv:1202.3665 [astro-ph.IM]]
Affine invariant MCMC
J.Goodman and J.Weare, CAMCSc 5, no.1, 65-80 (2010) doi:10.2140/camcos.2010.5.65

Posterior distributions

CORNELL POTENTIAL

Parameters	Potential I
α_s	$0.38^{+0.06}_{-0.03}$
σ (in GeV^2)	$0.19^{+0.01}_{-0.02}$
ρ (in GeV)	$2.28^{+0.80}_{-0.62}$
V_c (in GeV)	$-0.18^{+0.10}_{-0.05}$

- τ_{int} ~ 560 – 620 over 55000 steps per walker (32 walkers total)
- Flattened chain ~ 1.4 × 10^6 samples (after discarding initial 20%)
- ESS of ~ 2300
- Mean acceptance fraction ~0.3



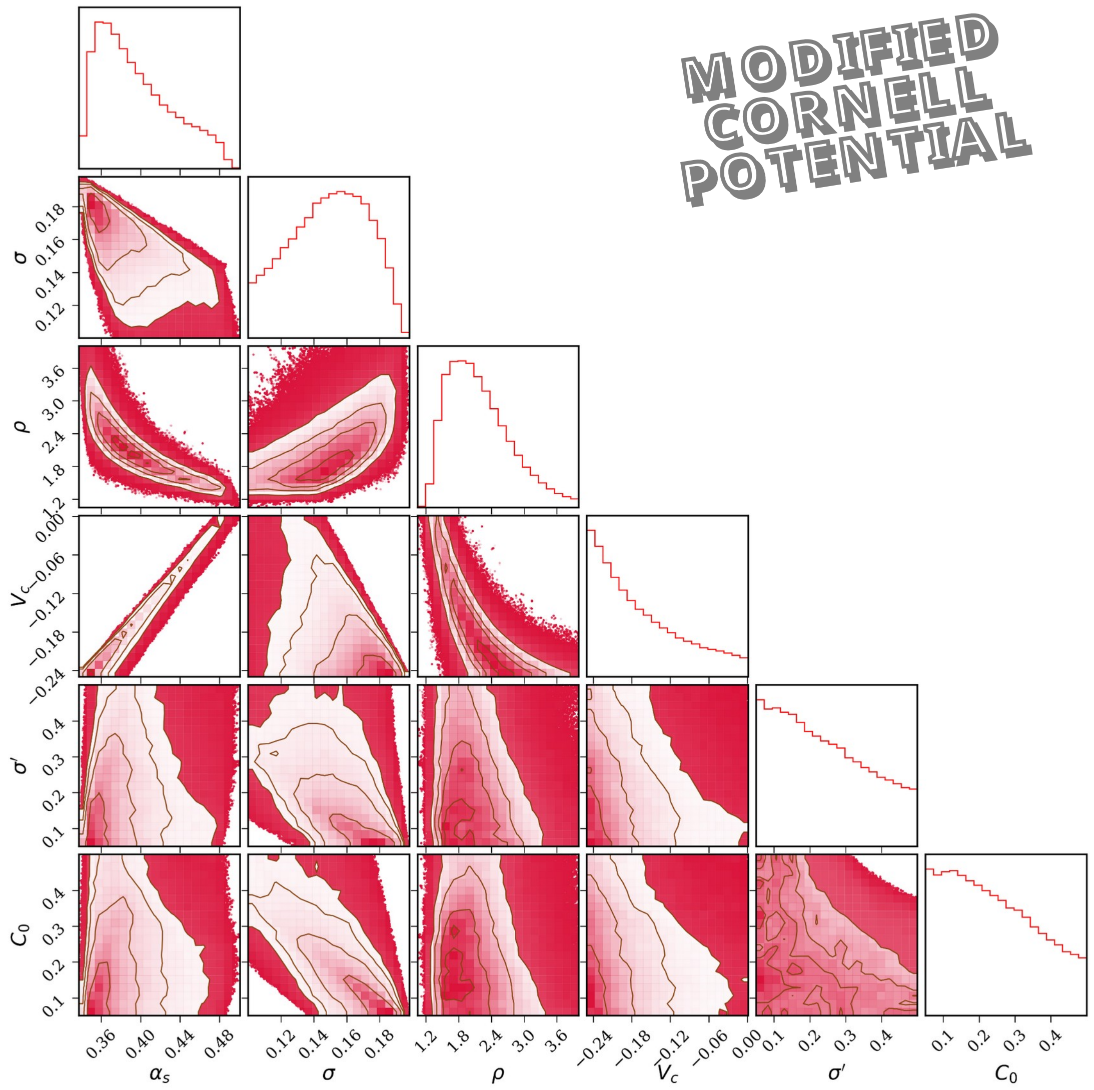
C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Posterior distributions

MODIFIED CORNELL POTENTIAL

Parameters	Potential I	Potential II
α_s	$0.38^{+0.06}_{-0.03}$	$0.39^{+0.05}_{-0.03}$
σ (in GeV^2)	$0.19^{+0.01}_{-0.02}$	$0.15^{+0.02}_{-0.03}$
ρ (in GeV)	$2.28^{+0.80}_{-0.62}$	$2.09^{+0.69}_{-0.48}$
V_c (in GeV)	$-0.18^{+0.10}_{-0.05}$	$-0.19^{+0.10}_{-0.05}$
σ' (in GeV)	-	$0.22^{+0.17}_{-0.12}$
C_0 (in GeV)	-	$0.23^{+0.16}_{-0.12}$

- τ_{int} = 3400 – 7200 over ~390000 steps per walker (48 walkers total)
- Flattened chain ~ 1.5×10^7 samples (after discarding initial 20%)
- ESS of ~ 2000–4300
- Mean acceptance fraction 0.09

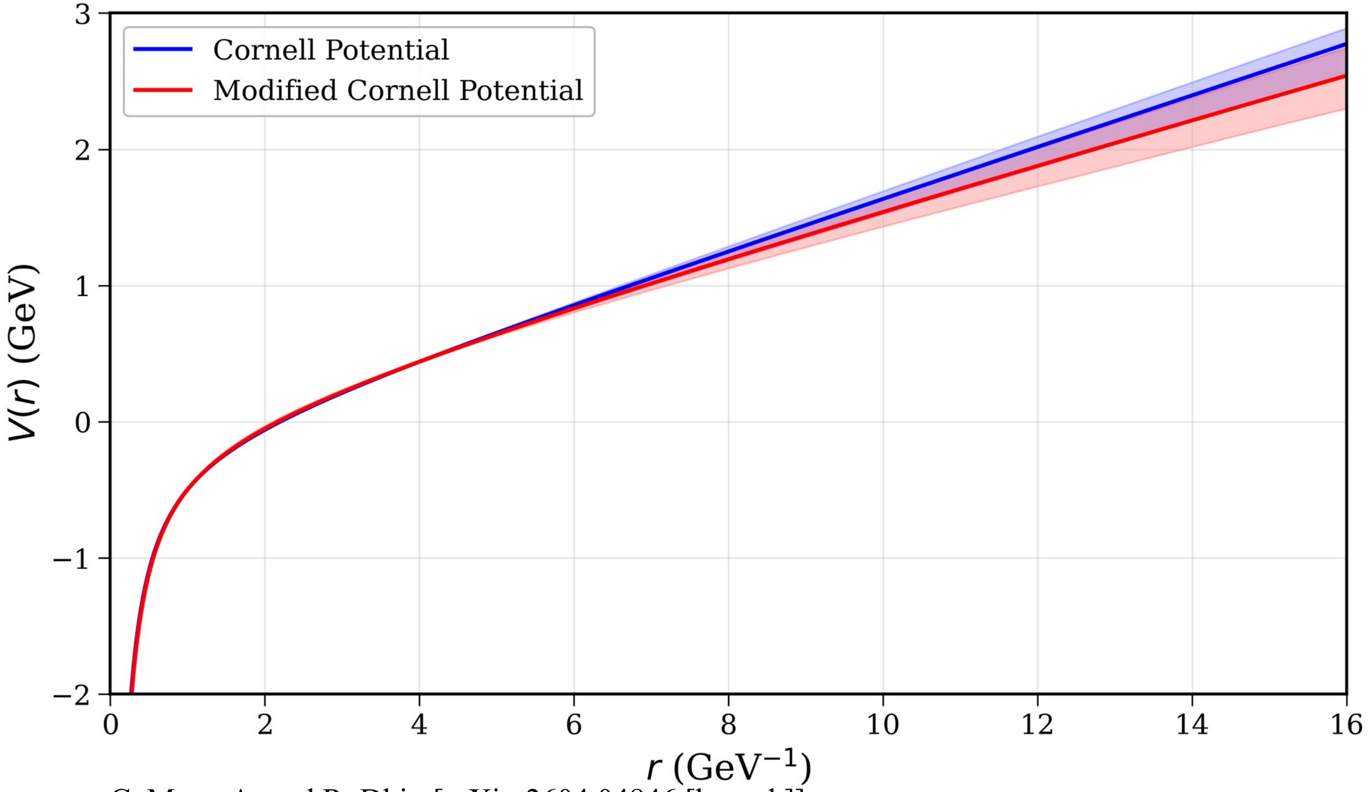


C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Why modify confinement?

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma r + V_c$$

$$V_{\text{Ext}}(r) = V(r) + C_0 \ln(1 + \sigma' r)$$



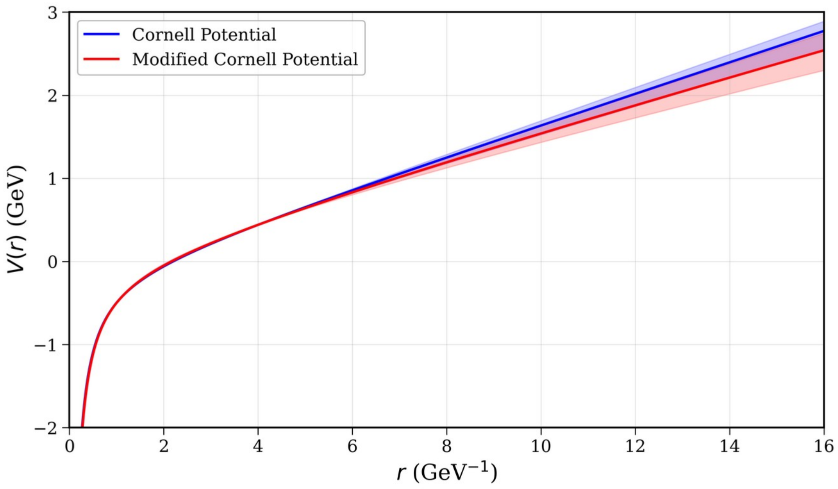
Parameters	Potential I	Potential II
α_s	$0.38^{+0.06}_{-0.03}$	$0.39^{+0.05}_{-0.03}$
σ (in GeV^2)	$0.19^{+0.01}_{-0.02}$	$0.15^{+0.02}_{-0.03}$
ρ (in GeV)	$2.28^{+0.80}_{-0.62}$	$2.09^{+0.69}_{-0.48}$
V_c (in GeV)	$-0.18^{+0.10}_{-0.05}$	$-0.19^{+0.10}_{-0.05}$
σ' (in GeV)	-	$0.22^{+0.17}_{-0.12}$
C_0 (in GeV)	-	$0.23^{+0.16}_{-0.12}$

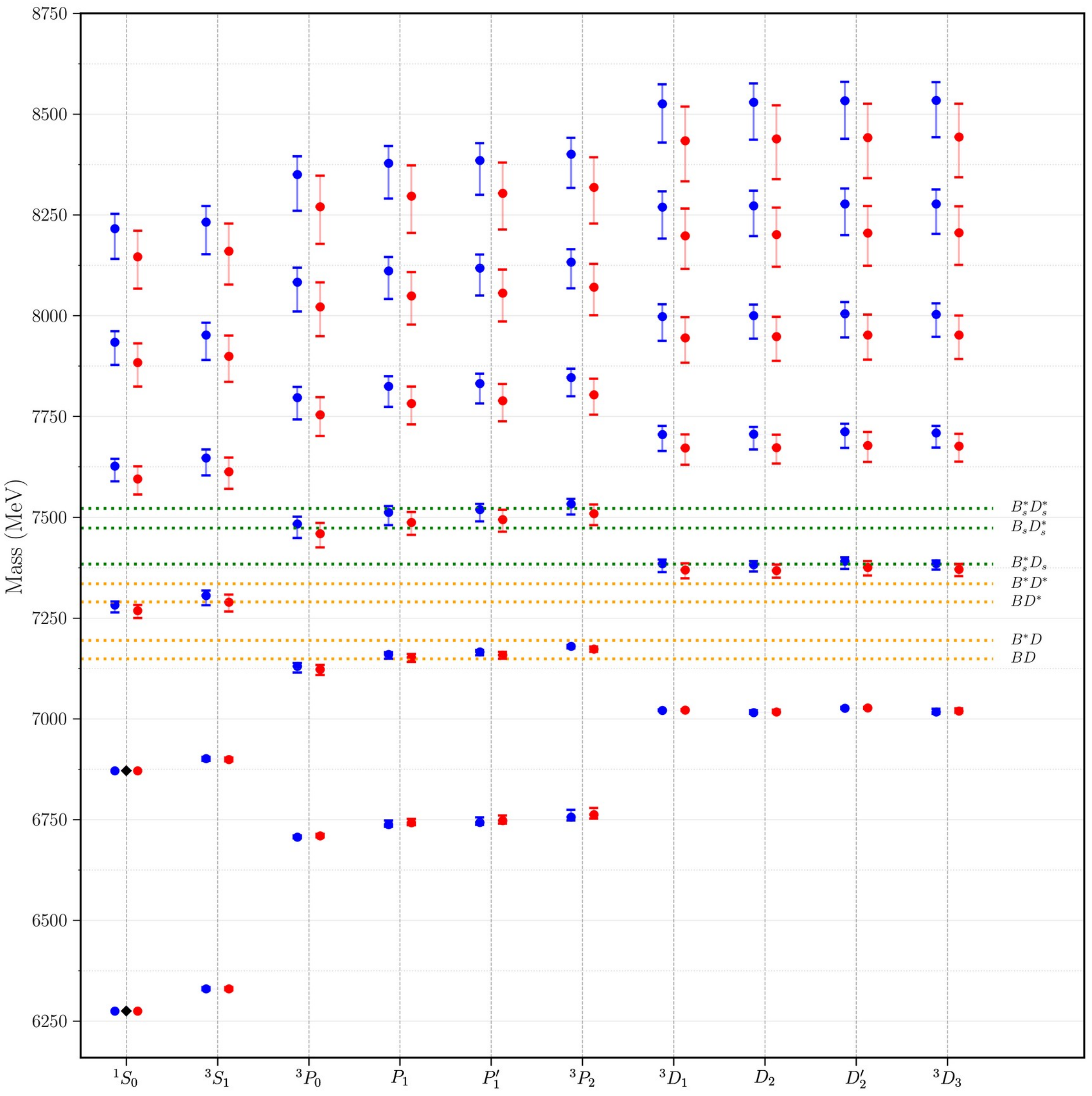
C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

B_c Spectra

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma r + V_c$$

$$V_{\text{Ext}}(r) = V(r) + C_0 \ln(1 + \sigma' r)$$





C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

State	Potential I	Potential II	PDG (Navas *et al.*, 2024)	LQCD (Mathur *et al.*, 2018)	LQCD (Dowdall *et al.*, 2012)
1^1S_0	6274^{+0}_{-0}	6274^{+0}_{-0}	$6274.47(0.32)^{\text{a}}$	6276(3)(6)	6278(9)
1^3S_1	6330^{+4}_{-4}	6330^{+4}_{-4}	-	6331(4)(6)	6332(9)
2^1S_0	6871^{+1}_{-1}	6871^{+1}_{-1}	$6871.2(1.0)^{\text{a}}$	-	6894(19)(8)
2^3S_1	6900^{+4}_{-5}	6898^{+5}_{-5}	-	-	6922(19)(8)
3^1S_0	7282^{+9}_{-19}	7268^{+15}_{-19}			
3^3S_1	7305^{+12}_{-24}	7289^{+18}_{-23}			
4^1S_0	7626^{+18}_{-38}	7595^{+31}_{-39}			
4^3S_1	7647^{+21}_{-43}	7612^{+35}_{-43}	-	-	-
5^1S_0	7933^{+27}_{-56}	7883^{+47}_{-59}	-	-	-
5^3S_1	7952^{+30}_{-62}	7899^{+51}_{-63}	-		-
6^1S_0	8216^{+36}_{-74}	8146^{+64}_{-79}	-	-	-
6^3S_1	8232^{+39}_{-80}	8160^{+68}_{-82}	-	-	-

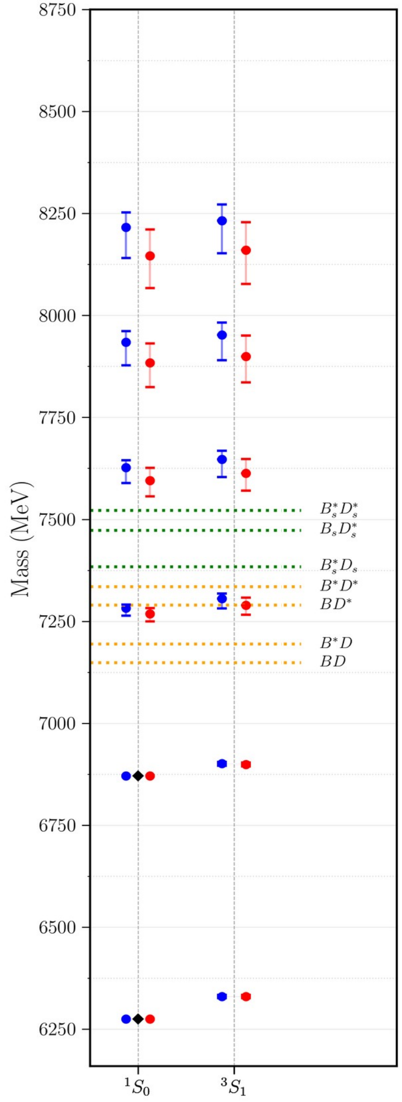
[a] Used as input, along with $M_{B_c^*(1S)} - M_{B_c(1S)} = 55(3)$ MeV from LQCD (Mathur *et al.*, 2018).

Bc* (arXiv:2605.16228 [hep-ex])

$64.5 \pm 1.4\ (\text{stat.})^{+1.0}_{-1.4}\ (\text{syst.})\ \text{MeV}$

$6339.0 \pm 1.4\ (\text{stat.})^{+1.0}_{-1.4}\ (\text{syst.}) \pm 0.3\ (m_{B_c^+})\ \text{MeV}$

$$\Delta M^{\text{I}}_{1S} = \Delta M^{\text{II}}_{1S} = 56^{+4}_{-4}\ \text{MeV}$$

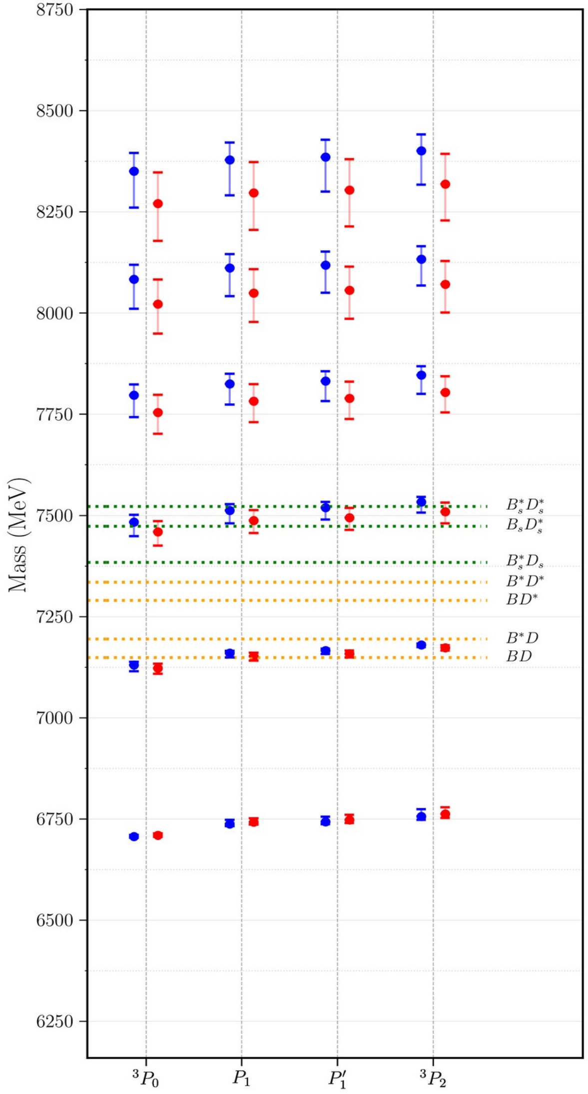


C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

State	Potential I	Potential II	LQCD (Mathur *et al.*, 2018)	LQCD (Dowdall *et al.*, 2012) (Davies *et al.*, 1996)
1^3P_0	6706^{+4}_{-3}	6709^{+4}_{-4}	6712(18)(7)	6707(14)(8)
$1P_1$	6737^{+9}_{-5}	6742^{+8}_{-6}	6736(17)(7)	6743(30)
$1P_1'$	6742^{+12}_{-6}	6747^{+11}_{-7}	-	6765(30)
1^3P_2	6756^{+17}_{-9}	6762^{+16}_{-10}	-	6783(30)
θ_{1P}	$(5^{+9}_{-10})^\circ$	$(7^{+8}_{-9})^\circ$	-	$33.4(1.5)^\circ$
2^3P_0	7130^{+7}_{-15}	7122^{+10}_{-14}		
$2P_1$	7159^{+5}_{-11}	7152^{+8}_{-10}		
$2P_1'$	7165^{+5}_{-8}	7158^{+7}_{-9}		
2^3P_2	7179^{+3}_{-4}	7173^{+5}_{-7}		
θ_{2P}	$(12^{+5}_{-5})^\circ$	$(13.36^{+5}_{-5})^\circ$		
3^3P_0	7483^{+17}_{-35}	7459^{+27}_{-34}	-	-
$3P_1$	7512^{+15}_{-31}	7487^{+25}_{-31}	-	-
$3P_1'$	7518^{+14}_{-29}	7494^{+24}_{-30}	-	-
3^3P_2	7533^{+12}_{-26}	7508^{+22}_{-28}	-	-
θ_{3P}	$(15^{+4}_{-4})^\circ$	$(16^{+4}_{-4})^\circ$	-	-

Orbitally excited states (Phys. Rev. Lett. 135 (2025) 231902, arXiv:2507.02149 [hep-ex])

$$M_1 = 6704.8 \pm 5.5 \pm 2.8 \pm 0.3 \ \text{MeV}/c^2$$
$$M_2 = 6752.4 \pm 9.5 \pm 3.1 \pm 0.3 \ \text{MeV}/c^2$$

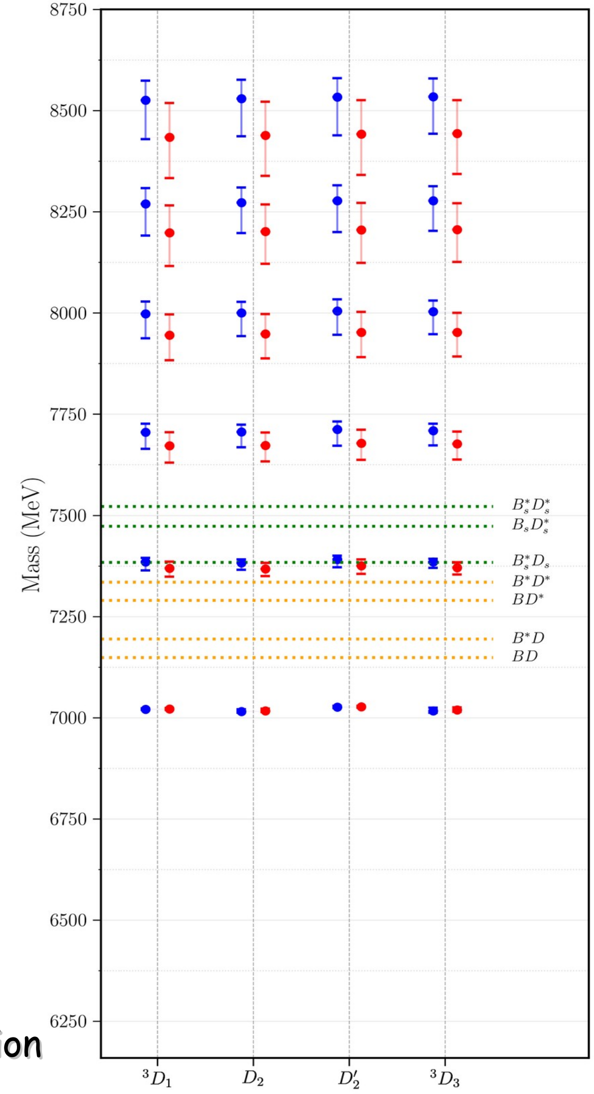


C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

State	Potential I	Potential II
1^3D_1	7021^{+2}_{-2}	7021^{+2}_{-2}
$1D_2$	7015^{+6}_{-4}	7017^{+5}_{-4}
$1D'_2$	7026^{+3}_{-3}	7027^{+3}_{-3}
1^3D_3	7016^{+7}_{-5}	7018^{+6}_{-4}
θ_{1D}	$(-53^{+2}_{-1})^\circ$	$(-52^{+2}_{-1})^\circ$
2^3D_1	7384^{+10}_{-21}	7369^{+16}_{-21}
$2D_2$	7382^{+8}_{-17}	7368^{+14}_{-18}
$2D'_2$	7390^{+9}_{-19}	7375^{+15}_{-20}
2^3D_3	7385^{+7}_{-15}	7370^{+13}_{-17}
θ_{2D}	$(-52^{+2}_{-1})^\circ$	$(-51^{+2}_{-1})^\circ$
3^3D_1	7705^{+20}_{-41}	7671^{+34}_{-42}
$3D_2$	7705^{+18}_{-38}	7672^{+32}_{-39}
$3D'_2$	7712^{+19}_{-40}	7678^{+32}_{-41}
3^3D_3	7709^{+17}_{-36}	7676^{+30}_{-38}
θ_{3D}	$(-51^{+2}_{-1})^\circ$	$(-50^{+2}_{-1})^\circ$

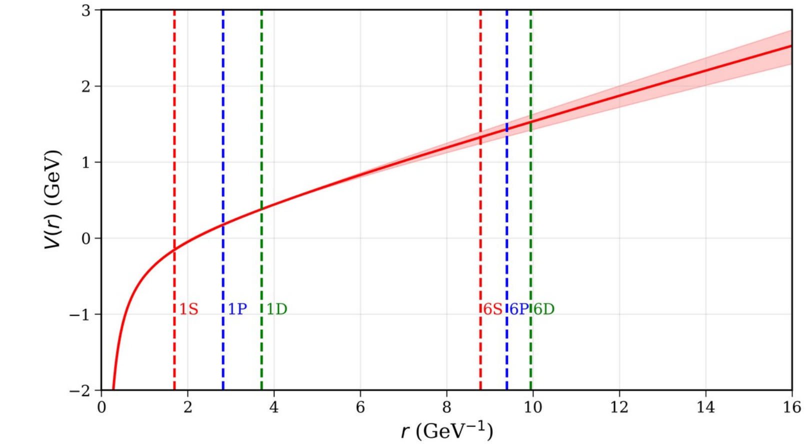
Mass ordering

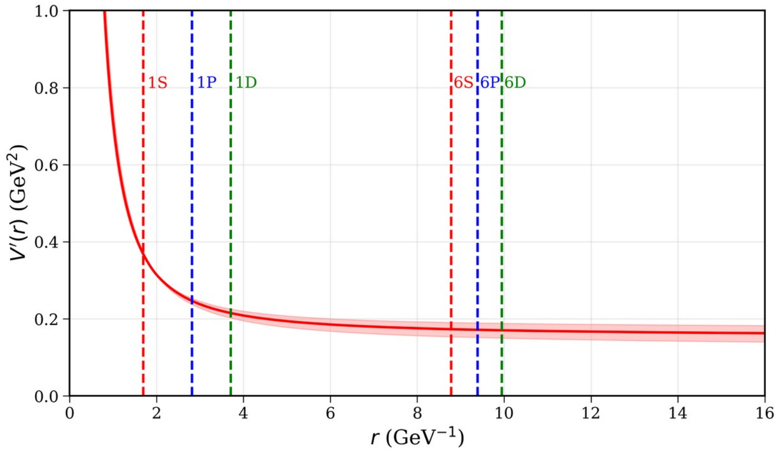
Mass separation grows with excitation



C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

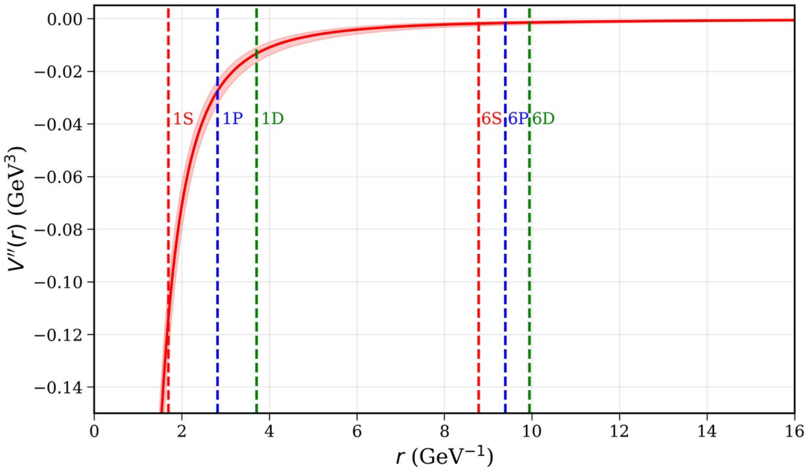
Effects of Modification in the Cornell Potential

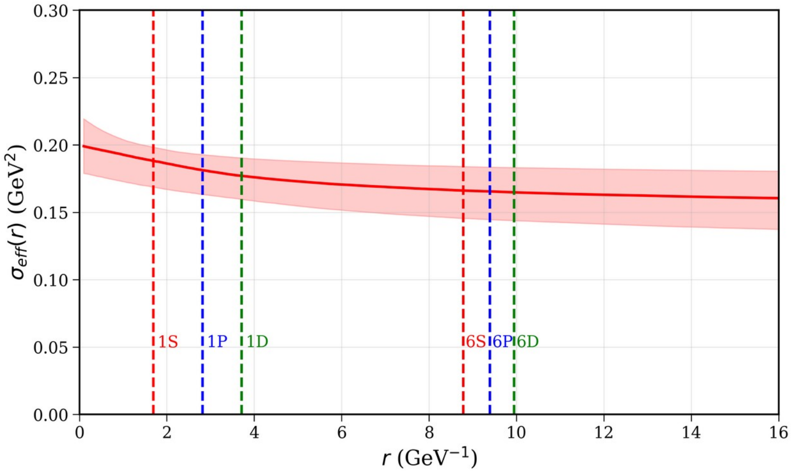




Effective string tension

$$\sigma_{\text{eff}}(r) = \sigma + \frac{C_0 \sigma'}{1 + \sigma' r}$$





$$\sqrt{\langle r^2 \rangle} = \sqrt{\int_0^\infty u_{nl}^*(r)\, r^2\, u_{nl}(r)\, dr}$$

Ground and excited states probe different regions

C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Closing thoughts

- Deterministic fits hide parameter degeneracies.
- Bayesian exploration reveal correlations & uncertainty.
- Excited B_c states probe confinement dynamics.
- Excited state multiplets are clear discriminators between confinement models.

Thank you!

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Spectroscopy of Heavy Quarkonium

SE

$$[T + V]\Psi = E\Psi$$

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma r + V_c$$

$$\Psi(\vec{r}) = R_{nl}(r) Y_{lm}(\theta, \phi)$$

WF

$$R_{nl}(r) = u_{nl}(r)/r$$

$$\int_0^\infty |R_{nl}(r)|^2 r^2 dr = \int_0^\infty |u_{nl}(r)|^2 dr = 1$$

SRWE

$$u''_{nl}(r) + 2\mu\left[E_{nl} - V(r) - \frac{l(l+1)}{2\mu r^2}\right] u_{nl}(r) = 0$$

Degenerate mass of the bound state

Runge-Kutta

$$M = m_Q + m_{\bar{Q}} + E_{nl}$$

$$R_{nl}^{(l)}(0) = \left.\frac{d^l R_{nl}(r)}{dr^l}\right|_{r=0}$$

$$\Psi(0) = \frac{R_{nS}(0)}{\sqrt{4\pi}}$$

Spin Dependent Interactions

$$V_{SS}(r) = \frac{2}{3m_1m_2}\mathbf{S}_1\cdot\mathbf{S}_2\,\Delta V_V(r)$$

$$V_T(r) = \frac{1}{12m_1m_2}S_{12}\left[\frac{1}{r}\frac{d}{dr}V_V(r) - \frac{d^2}{dr^2}V_V(r)\right]$$

$$S_{12} \equiv 12\left[\frac{(\mathbf{S}_1\cdot\mathbf{r})(\mathbf{S}_2\cdot\mathbf{r})}{r^2} - \frac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2\right]$$

$$V_{LS}(r) = \frac{1}{4m_1^2m_2^2r}\left\{\left[\left((m_1+m_2)^2+2m_1m_2\right)\mathbf{L}\cdot\mathbf{S}_+ + (m_2^2-m_1^2)\mathbf{L}\cdot\mathbf{S}_-\right]\frac{d}{dr}V_V(r) - \left[(m_1^2+m_2^2)\mathbf{L}\cdot\mathbf{S}_+ + (m_2^2-m_1^2)\mathbf{L}\cdot\mathbf{S}_-\right]\frac{d}{dr}V_S(r)\right\}$$

Mixing*

$$M(n^{2S+1}L_J) = m_Q + m_{\bar{Q}} + E_{nl} + \langle u_{nl}|V_{SS}|u_{nl}\rangle + \langle u_{nl}|V_{LS}|u_{nl}\rangle + \langle u_{nl}|V_T|u_{nl}\rangle$$

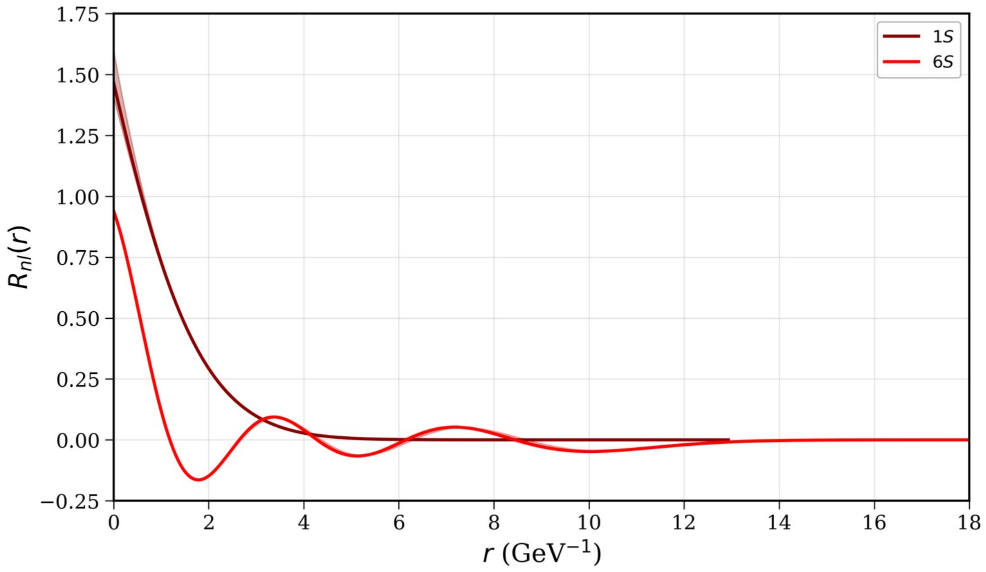
Eichten E et al 1975 PRL 34 369–72
Eichten E et al 1976 PRL 36 1276
Eichten E et al 1980 PRD 21 203

Eichten E & Feinberg F L 1979 PRL 43 1205
Eichten E & Feinberg F 1981 PRD 23 2724
Gromes D 1984 ZPC 26 401

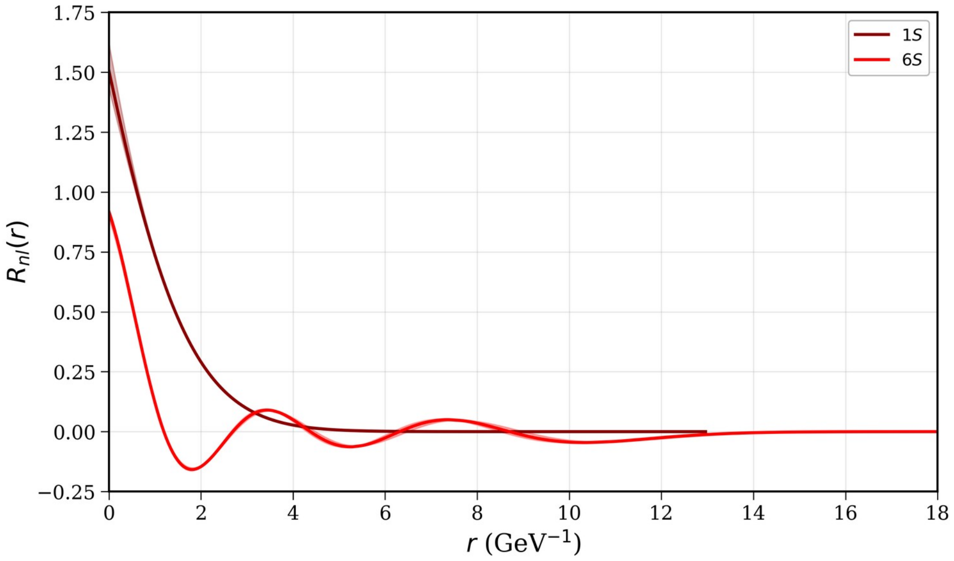
Lucha W & Schoberl F F arXiv:hep-ph/9601263
Barnes T et al 2005 PRD 72 054026
Ikhdair S M & Sever R 2005 IJMPA 20 4035-4054

Radial wave functions (S-wave)

CORNELL POTENTIAL



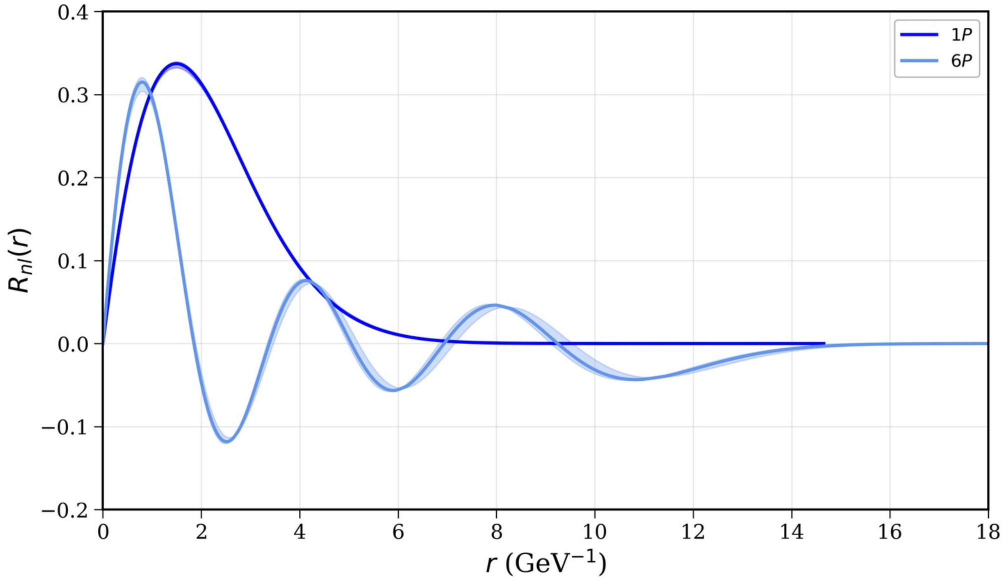
MODIFIED CORNELL POTENTIAL



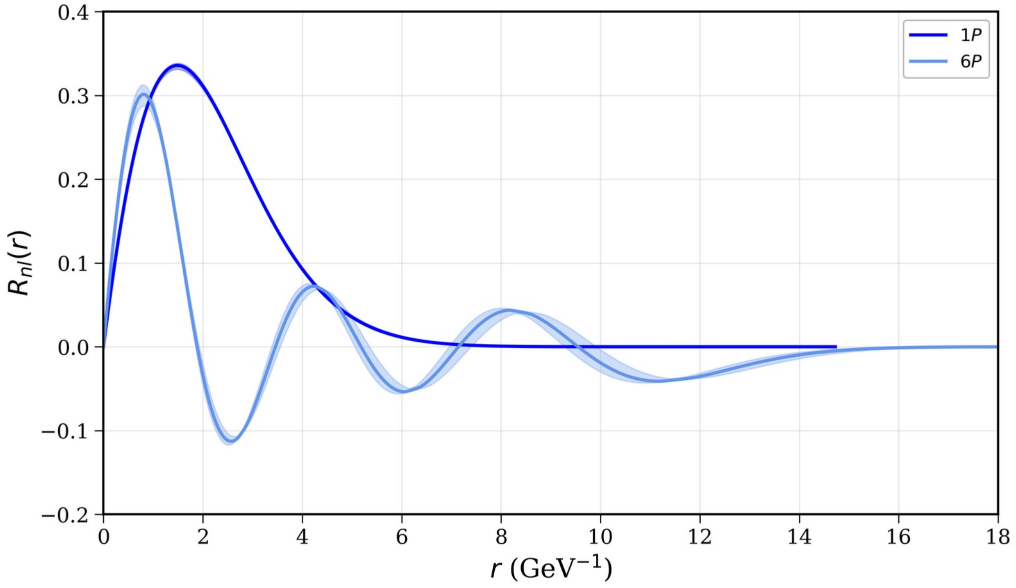
C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Radial wave functions (P-wave)

CORNELL POTENTIAL



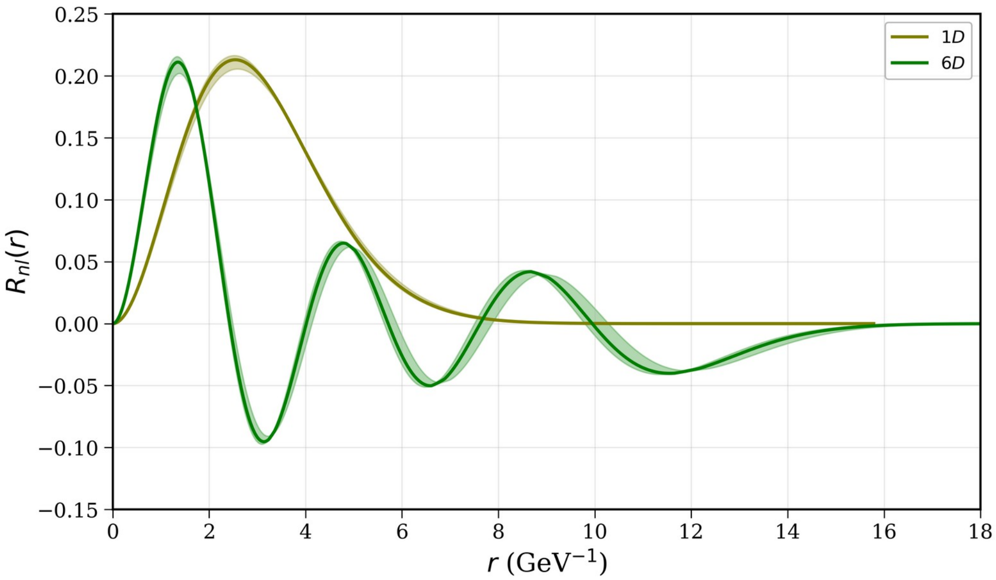
MODIFIED CORNELL POTENTIAL



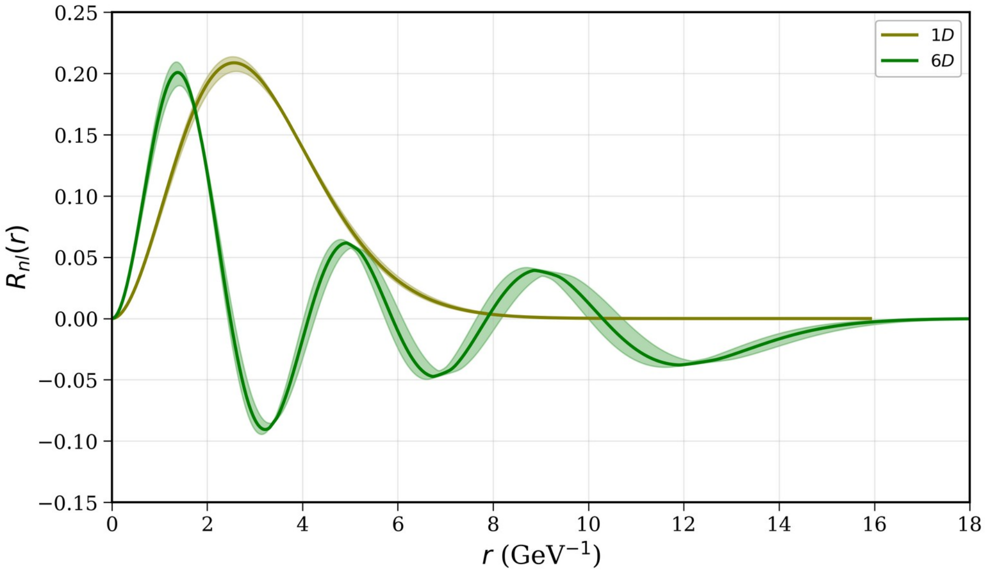
C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].

Radial wave functions (D-wave)

CORNELL POTENTIAL



MODIFIED CORNELL POTENTIAL



C. Mony A. and R. Dhir, [arXiv:2604.04846 [hep-ph]].