QCD Sum Rules and Their Applications to Hadron Spectroscopy and Decays

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Outline

The Problem: QCD is Hard at Low Energies

- QCD is the accepted theory of the strong interaction, with quarks and gluons as fundamental degrees of freedom.
- At high energy (large Q^2), asymptotic freedom makes perturbation theory reliable.
- At low energy (hadronic scale, ~ 1 GeV), α_s is large $\Rightarrow$ perturbation theory fails.
- Yet this is exactly the regime where hadrons (mesons, baryons, exotics) live!

The central question

How do we connect quark-gluon language (QCD Lagrangian) to hadron language (masses, decay constants, form factors) *without* solving QCD exactly?

Tools for the Nonperturbative Regime

- **Lattice QCD** – first-principles, but computationally expensive; historically difficult for some exotic/multi-quark channels and for excited states.
- **Quark models / constituent models** – intuitive, but parameters are fit, not derived from QCD.
- **Effective field theories** (ChPT, HQET, NRQCD) – systematic, but limited regime of validity.
- **QCD Sum Rules (QCDSR)** – analytic, directly built on the QCD Lagrangian via the operator product expansion (OPE); modest computational cost.

QCDSR sit in a useful middle ground: they are *semi-analytic*, model-independent in their assumptions (mostly), and applicable to a huge variety of channels — including exotic hadrons.

What Is a QCD Sum Rule, in One Slide?

1. Choose an interpolating current $j(x)$ with the quantum numbers of the hadron of interest.
2. Compute the two-point correlator $\Pi(q^2) = i \int d^4x\, e^{iqx} \langle 0 | T\{j(x) j(0)\} | 0 \rangle$ in the deep Euclidean region using the operator product expansion (perturbative part + vacuum condensates).
3. Relate the *same* correlator to the hadronic spectrum via a dispersion relation.
4. Match the two representations (quark-hadron duality) using a Borel transform to suppress excited-state contamination.
5. Extract hadron mass, coupling, or decay constant by demanding stability against the unphysical Borel parameter.

Why This Talk Is Structured This Way

- QCDSR can look like a wall of algebra on first exposure. The *physics* is simple: it is built on completely standard ideas from quantum mechanics.
- So we will first build every concept — Green's functions, dispersion relations, Borel transforms, duality — in an exactly solvable toy model: the quantum harmonic oscillator.
- Only after that do we write down the QCD version, discuss trust & limitations, present an application (the B^*K^* molecule), and finish with modern extensions (e.g. light-cone sum rules).

Why a Quantum-Mechanical Toy Model?

- QCD sum rules were invented by Shifman, Vainshtein and Zakharov (SVZ, 1979).
- Every key ingredient of the method — condensates, OPE, dispersion relations, Borel improvement, duality — already appears in ordinary quantum mechanics for a particle in a confining potential.
- The 2D harmonic oscillator is exactly solvable, so we can compare the sum-rule *estimate* of the ground-state energy directly against the *exact* answer.
- This builds the intuition needed to trust (and to doubt, appropriately) the QCD calculation later.

Setup: The 2D Oscillator

Confining potential (we work in $2+1$ dimensions for algebraic simplicity):

$$V(\vec{x}) = \frac{m\omega^2}{2} r^2.$$

Exact energy levels and wavefunctions at the origin:

$$E_n = (2n+1)\,\omega, \qquad |\psi_n(0)|^2 = \frac{m\omega}{\pi}.$$

- ω plays the role of the "confinement scale" (analogous to Λ_{QCD}).
- $\psi_n(0)$ is the analogue of a hadronic decay constant such as f_π.
- Goal: recover E_0 and $\psi_0(0)$ *without* solving the Schrödinger equation directly — mimicking what we must do in QCD, where exact wavefunctions are unknown.

The Two-Point Function: A Sum Over All States

Instead of isolating the ground state directly, sum over *all* levels with an exponential weight:

$$M(\epsilon) = \sum_{k=0}^{\infty} |\psi_k(0)|^2 \, e^{-E_k/\epsilon}.$$

- For $\epsilon \to 0$ only the ground state survives.
- For large ϵ, many states contribute — but this is exactly where the calculation is easiest!
- $M(\epsilon)$ is the imaginary-time analogue of the Green's function $G(x_1, t_1|x_2, t_2)$, evaluated at coincident points with $t_2 - t_1 = 1/(i\epsilon)$.

This object, computable in two completely different ways, is the seed of every sum rule.

Exact Result and the Free-Particle Limit

For the oscillator,

$$M^{osc}(\epsilon) = \frac{m\omega}{\pi \sinh(\omega/\epsilon)} \xrightarrow{\text{large } \epsilon} \frac{m\epsilon}{2\pi}\left[1 - \frac{\omega^2}{6\epsilon^2} + \frac{7}{360}\frac{\omega^4}{\epsilon^4} - \cdots\right].$$

- The leading term, $m\epsilon/(2\pi)$, has no ω dependence: at very short (imaginary) times the particle does not yet "feel" the confining potential.
- This is the free-particle (perturbative) result,

$$M^{free}(\epsilon) = \frac{m\epsilon}{2\pi}.$$

- Corrections in powers of ω^2 encode the effect of confinement — the direct analogue of power corrections / condensates in QCD.

Spectral Densities and Global Duality

Both M^{free} and M^{osc} can be written as $\frac{1}{\pi}\int_0^\infty e^{-E/\epsilon}\rho(E)\,dE$ with

$$\rho^{free}(E) = \frac{m}{2}\,\theta(E > 0), \qquad \rho^{osc}(E) = m\omega\sum_{k=0}^{\infty}\delta(E - (2k+1)\omega).$$

- These look nothing alike pointwise (a smooth curve vs. a comb of delta functions)...
- ...yet *integrated* over any energy window of width 2ω, the two densities agree exactly:

$$\int_0^\infty \left[\rho^{osc}(E) - \rho^{free}(E)\right] dE = 0.$$

This is global quark–hadron duality: free (perturbative) and resonance spectra agree *on average*, even though they differ resonance-by-resonance.

Building the Sum Rule

Ansatz: replace the unknown "higher states" contribution by the free spectral density above an effective threshold s_0:

$$\rho^{higher}(E) = \left[1 - \theta(E < s_0)\right] \rho^{free}(E).$$

This gives the sum rule

$$|\tilde{\psi}_0(0)|^2 e^{-E_0/\epsilon} = \frac{\epsilon}{2}\left(1 - e^{-s_0/\epsilon}\right) - \frac{\omega^2}{12\epsilon} + \frac{7}{720}\frac{\omega^4}{\epsilon^3} - \cdots$$

- Left side: "ground state" contribution (unknown E_0, $\psi_0(0)$).
- Right side: free spectral density truncated at s_0, plus a few power corrections – all computable.
- Three unknowns to fit: E_0, $\tilde{\psi}_0(0)$, and the threshold s_0.

Extracting E_0: Differentiate and Take Ratios

Differentiating with respect to $(-1/\epsilon)$ and dividing the two sum rules:

$$E_0 = \frac{\dfrac{\epsilon^2}{2} - \dfrac{\epsilon^2}{2}\left(1 + \frac{s_0}{\epsilon}\right) e^{-s_0/\epsilon} + \dfrac{\omega^2}{12} - \cdots}{\dfrac{\epsilon}{2}\left(1 - e^{-s_0/\epsilon}\right) - \dfrac{\omega^2}{12\epsilon} + \cdots}.$$

- If we kept the *entire* series and the exact higher-state contribution, $E_0 = \omega$ exactly, independent of ϵ.
- With our truncated, approximate ansatz, E_0 depends (weakly) on ϵ and on s_0.
- **Strategy:** choose s_0 so that $E_0(\epsilon)$ is as *flat* as possible over an intermediate "Borel window" $1 < \epsilon/\omega < 2$.

Results: How Good Is the Estimate?

Truncation order	s_0	E_0 (extracted)	E_0 (exact)
2nd order in ω^2	$1.6\,\omega$	$0.90\,\omega$	ω
3rd order in ω^2	$1.75\,\omega$	$0.95\,\omega$	ω

- Even with a crude 2–3 term truncation and a schematic "free states above s_0" model, we recover E_0 to within 5–10%.
- The dominant error comes from modeling the *excited-state* contribution, not from the power corrections themselves.
- This is the precision QCDSR practitioners quote for real hadrons: $\sim$ 10–20%.

Local Duality and the Form Factor

The same logic extends to *form factors* $F_{kl}(Q^2) = \int \psi_k^*(x)\psi_l(x)e^{iQx}d^2x$, using a doubled Green's function (one for each external leg).

- The exact ground-state form factor is Gaussian: $F_{00}(Q^2) = \exp(-Q^2/4m\omega)$.
- The "local duality" model — replacing the bound-state wavefunction by a sharp momentum-space step at $k^2 < 2ms_0$ — reproduces the correct normalization and overall shape, but gets the *slope* at $Q^2 = 0$ slightly wrong.
- Lesson: duality works well for *integrated* quantities (masses, decay constants) but needs more care for differential observables (form factors, especially near threshold).

From Quantum Mechanics to QCD: The Dictionary

Oscillator quantity	QCD analogue
Confinement scale ω	Λ_{QCD}, condensate scales
Green's function $M(\epsilon)$	Correlator $\Pi(q^2)$
Imaginary-time weight $e^{-E/\epsilon}$	Borel transform e^{-s/M^2}
$\psi_n(0)$	Decay constant (f_π, etc.)
Free spectral density	Perturbative spectral density
$\omega^2, \omega^4, \ldots$ corrections	Quark/gluon condensates
"Free states above s_0" ansatz	"Resonance + continuum" ansatz

Every step we now take in QCD has a direct counterpart above.

Correlators in Quantum Field Theory

The QFT analogue of the Green's function is the two-point correlator of a hadronic interpolating current $j(x)$:

$$\Pi(q) = i \int d^4x\, e^{iqx}\, \langle 0 | T\{j(x) j(0)\} | 0 \rangle.$$

- For a ρ-meson: $j_\mu = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$.
- For a pion: $j^5_\alpha = \bar{d}\gamma_5\gamma_\alpha u$.
- In perturbation theory $\Pi(q^2)$ is computable for spacelike, large $|q^2|$ — exactly where asymptotic freedom holds.

Operator Product Expansion (OPE)

The exact quark/gluon propagator differs from the free one because the QCD vacuum is *not* the perturbative vacuum:

$$\langle 0|T\{\varphi(x)\varphi(0)\}|0\rangle \;\neq\; \langle \Omega|T\{\varphi(x)\varphi(0)\}|\Omega\rangle.$$

Taylor-expanding the difference at short distance generates local operators of increasing dimension:

$$\Pi(Q^2) = \Pi^{pert}(Q^2) + \frac{c_3}{Q^4}\langle \bar{q}q\rangle + \frac{c_4}{Q^4}\langle G^2\rangle + \frac{c_6}{Q^6}\langle \bar{q}q\rangle^2 + \cdots$$

- $\langle \bar{q}q\rangle$: quark condensate (dim. 3); $\langle G^2\rangle$: gluon condensate (dim. 4); four-quark condensate (dim. 6); mixed condensate $\langle \bar{q}g_s\sigma\cdot Gq\rangle$ (dim. 5), etc.
- These are *universal, channel-independent* vacuum parameters — fixed once, used everywhere.

Dispersion Relation: Connecting to Hadrons

The same $\Pi(q^2)$ also satisfies an unsubtracted (or subtracted) dispersion relation:

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \frac{\rho(s)}{s - q^2}\, ds + (\text{subtractions}),$$

where the spectral density $\rho(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)$ contains *all* physical information: pole(s) for stable states plus a continuum for multi-particle states.

Two representations, one function

- OPE side: computable for $Q^2 = -q^2 \gg 0$ (quark-gluon language).
- Dispersive side: built from physical hadrons (mass, coupling, ...).
- Equate them $\Rightarrow$ a sum rule.

The Borel Transform

Direct matching of the two sides is numerically unstable: power corrections in $1/Q^2$ converge slowly, and continuum/excited states are not exponentially suppressed.
The Borel transform,

$$\Phi(M^2) \equiv \lim_{Q^2, n\to\infty,\ Q^2/n=M^2} \frac{(Q^2)^n}{(n-1)!} \left(-\frac{d}{dQ^2}\right)^n \Pi(Q^2),$$

acting term by term gives

$$B\left[\frac{1}{(Q^2)^n}\right] = \frac{1}{(n-1)!\,(M^2)^n}, \qquad B\left[\frac{1}{s+Q^2}\right] = \frac{1}{M^2} e^{-s/M^2}.$$

Two simultaneous benefits: (i) power corrections are factorially improved; (ii) the dispersion integral acquires the exponential weight e^{-s/M^2} — exactly the ϵ-weighting of the oscillator problem, now with Borel mass M in place of ϵ.

Quark–Hadron Duality, Again

After the Borel transform, the master sum rule reads schematically

$$\underbrace{f_H^2\, e^{-m_H^2/M^2}}_{\text{ground state}} + \underbrace{\int_{s_0}^{\infty} \rho^{pert}(s)\, e^{-s/M^2}\, ds}_{\text{continuum (duality ansatz)}} = \underbrace{\Pi^{OPE}(M^2)}_{\text{perturbative + condensates}} .$$

- Same “resonance + continuum” ansatz as in the oscillator problem.
- s_0 is the effective continuum threshold — a fit parameter, not a fundamental input.
- One demands a Borel window: M^2 large enough that the OPE (truncated at some condensate dimension) converges, but small enough that the ground state dominates over the continuum.

Worked Example: The ρ-Meson (SVZ)

For the isovector vector current, the Borel-transformed sum rule (schematically, with standard condensate values) gives

$$\int_0^\infty e^{-s/M^2} R^{I=1}(s)\, ds = \frac{3}{2} M^2 \Big[1 + \frac{\alpha_s}{\pi} + 0.1 \Big(\frac{0.6\,\mathrm{GeV}^2}{M^2} \Big)^2 - 0.14 \Big(\frac{0.6\,\mathrm{GeV}^2}{M^2} \Big)^3 \Big].$$

Using a "resonance + continuum" ansatz and a stability analysis in M^2 and s_0:

$$s_0 \approx 1.5\ \mathrm{GeV}^2, \qquad m_\rho^2 \approx 0.58\text{–}0.60\ \mathrm{GeV}^2, \qquad g_\rho^2 \approx 2.2\text{–}2.4,$$

to be compared with experiment, $m_\rho \approx 0.77$ GeV and $g_\rho^2/4\pi = 2.36 \pm 0.18$. Agreement at the $\lesssim 10\%$ level – the hallmark QCDSR result.

Pion Channel: A Special, Very Clean Case

- The axial current correlator splits into transverse (Π_1) and longitudinal (Π_2) pieces.
- In the chiral limit, the famous relation

$$f_\pi^2 m_\pi^2 = -(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle$$

  follows directly from the OPE + dispersion relation, demonstrating that $m_\pi \to 0$ as $m_q \to 0$ – the pion as a (pseudo-)Goldstone boson, derived from QCDSR itself.
- A simple sum rule analysis gives $4\pi^2 f_\pi^2 \approx 0.65$–$0.69$ GeV^2 vs. the experimental 0.67 GeV^2.

Pion Electromagnetic Form Factor

Going beyond masses, the *same* machinery applied to a three-point function $\langle j_\beta \, J^\mu \, j_\alpha^+ \rangle$ yields the pion form factor $F_\pi(Q^2)$.

- QCDSR predictions match data at the 10–20% level across $0.5 \lesssim Q^2 \lesssim 3$ GeV^2.
- The extracted pion charge radius, $\langle r_\pi^2 \rangle^{1/2} \approx 0.66 \pm 0.03$ fm, agrees with the experimental 0.636 ± 0.036 fm.
- Key qualitative result: in the experimentally accessible region, the form factor is dominated by the *soft* (nonperturbative) contribution, not by the perturbative one-hard-gluon-exchange mechanism.

Sources of Theoretical Uncertainty

- **OPE truncation** – only the first few condensates are ever known/included; higher-dimension condensates are increasingly poorly determined.
- **Continuum/duality ansatz** – "ground state + free continuum above s_0" is a model, not a theorem; the true spectral density also has structure just above threshold.
- **Condensate values** – some (e.g. $\langle \bar{q}q \rangle$, $\langle G^2 \rangle$) are well constrained from light-hadron phenomenology; four-quark condensates rely on the factorization (vacuum saturation) hypothesis, itself an approximation.
- α_s **and quark mass scheme dependence** at higher loop order.
- **Borel window** must exist and be wide enough; for some channels (especially multiquark/exotic states) this window can be narrow or ambiguous.

So... How Much Can We Trust the Numbers?

- For well-studied light-quark channels ($\rho, \pi, N, \ldots$): typical accuracy 10–20%, matching the oscillator toy-model error budget exactly.
- For heavy-quark and exotic channels: uncertainties are often larger and harder to quantify, because:
  - fewer independent sum rules to cross-check;
  - less experimental data to anchor s_0;
  - condensates of higher dimension matter more (heavy-quark expansion pushes power corrections to higher order, but each one is less well known).
- **Best practice:** always perform a Borel-stability analysis, vary s_0, and quote results as *ranges*, not single numbers.

Limitations: A Short, Honest List

1. Cannot predict the *existence* of a resonance from first principles – it tells you whether a postulated current/state is *consistent* with QCD, given an ansatz for the spectrum.
2. Weak sensitivity to fine spectral details (widths, exact spin/parity assignments in degenerate channels).
3. Difficulty with channels that mix strongly with nearby thresholds or have large widths (relevant for many exotic candidates!).
4. Results depend on a hierarchy of approximations (OPE truncation + duality ansatz + condensate values) whose *combined* error is hard to propagate rigorously.

Why QCDSR Are Still Used Today

- **Flexibility**: any current with the right quantum numbers can be analyzed – ideal for the zoo of exotic hadrons (tetraquarks, pentaquarks, hadronic molecules, hybrids) discovered since the 2000s.
- **Speed**: a sum rule calculation for a new multiquark current can be done in days/weeks, vs. months/years for dedicated lattice studies of the same channel.
- **Complementarity**: provides an independent cross-check to lattice QCD and quark models, especially useful before lattice results are available, or for channels lattice still struggles with.
- **Spectroscopic discrimination**: QCDSR can compare different structural assumptions for the *same* observed state (e.g. compact tetraquark vs. hadronic molecule) by simply changing the interpolating current.

B^*K^*

- Exotic candidates near two-meson thresholds are natural targets for a hadronic molecule interpretation: a loosely bound state of two color-singlet mesons, held together by long-range meson exchange.
- The open-flavor B^*K^* system, with quantum numbers $J^P = 0^+$, is a natural candidate to probe with QCDSR using a heavy-light current.
- This section presents a QCDSR-style calculation of the mass of a B^*K^* molecular state, performed with a FeynCalc-automated OPE up to dimension 5.

Interpolating Current and Correlator

- A $J^P = 0^+$ B^*K^* molecular current is built from heavy (b) and light ($u/d, s$) quark fields, with the structure of a color-singlet B^*–K^* pair.
- The relevant object is the two-point correlator

$$\Pi(p_1) = i \int d^4x\, e^{ip_1x} \langle 0| T\{j(x) j^\dagger(0)\}|0\rangle,$$

  computed in coordinate (x-)space using exact propagators for the heavy (b) quark and the light (u/d, s) quarks, including condensate insertions.
- The light-quark propagator is expanded to include the quark condensate $\langle \bar{q}q \rangle$, gluon condensate $\langle G^2 \rangle$, and mixed condensate $\langle \bar{q} g_s \sigma \cdot G q \rangle$ terms – dimensions 0, 3, 4, and 5 of the OPE.

Phase 1: Automated OPE Construction

- The Dirac trace algebra for the heavy-light correlator was performed symbolically in `FeynCalc` (Mathematica).
- The propagators are organized as

$$S_{light}(x) = \frac{i}{2\pi^2}\frac{\not{x}}{x^4} - \frac{m_q}{4\pi^2 x^2} - \frac{1}{12}\langle\bar{q}q\rangle + \cdots$$

  with condensate insertions carrying explicit ϵ-bookkeeping markers, used purely to *sort* the resulting trace by operator dimension.
- Output: four separate analytic expressions, $\Pi^{(0)}, \Pi^{(3)}, \Pi^{(4)}, \Pi^{(5)}$, corresponding to the perturbative term and the $\langle\bar{q}q\rangle$, $\langle G^2\rangle$, $\langle\bar{q}Gq\rangle$ condensate contributions.

Model Construction: A Phenomenological Ansatz

The full analytic continuation of an x-space OPE to a momentum-space spectral density for a multi-quark current is technically demanding. We therefore present this calculation explicitly as a phenomenological ansatz for the spectral density in the heavy-quark limit, built on three physically motivated approximations.

Bullet summary

- **OPE**: exact x-space correlator via FeynCalc, up to dimension 5.
- **Kinematics**: HQET rest-frame projection for the heavy b-quark.
- **Spectral density**: threshold phase-space mapping $\rho(s) \propto (s - s_{min})^k$.
- **Convergence**: effective condensate weights stand in for unevaluated $\gtrsim$Dim-8 terms.

Justification 1: Threshold Phase-Space Dominance

Assumption: map coordinate-space singularities $1/x^n$ directly onto threshold power-law scaling $(s - s_{min})^k$ in momentum space, instead of carrying out the full off-shell Fourier transform.

- Hadronic molecules are, by construction, loosely bound states sitting very close to the two-meson kinematic threshold $s \approx s_{min}$.
- By the Cutkosky rules, near threshold the imaginary part of a diagram (the spectral density) is dominated by the *phase-space volume* of the final-state particles, which for an S-wave n-body state scales as a definite power of $(s - s_{min})$.
- This is therefore a standard, well-motivated way to probe the viability of a near-threshold bound state without evaluating the complete off-shell momentum integral.

Justification 2: HQET Kinematic Projection

Assumption: replace the exact tensor contraction $p_1 \cdot x$ by a scalar projection $\propto m_b$.

- The b-quark mass ($m_b \approx 4.18$ GeV) vastly exceeds the confinement scale; in its rest frame $p_1^\mu \approx m_b v^\mu$ to leading order in the HQET power counting.
- This reduces $p_1 \cdot x \to m_b x_0$.
- Because the target state has $J^P = 0^+$, the relative motion of the light degrees of freedom around the heavy quark is S-wave (isotropic); angular averaging over spatial coordinates then isolates exactly the temporal component used above.

This is the same heavy-quark/light-quark factorization discussed earlier (Sec. 3.3 of the formalism, $m \to \infty$ limit) – here applied directly to a multi-quark current.

Justification 3: Effective Condensates

Assumption: the dimension-4 and dimension-5 condensate *weights* in the numerical model are adjusted away from their bare textbook values.

- The OPE here is truncated at dimension 5; for heavy-light multiquark currents the heavy-quark mass suppresses the lowest dimensions, shifting sensitivity toward higher-dimension condensates (Dim-8, Dim-10, e.g. $\langle \bar{q}q \rangle \langle \bar{q} g_s \sigma \cdot G q \rangle$).
- Evaluating these multi-gluon topologies analytically is beyond the scope of the present framework.
- We instead introduce *effective* Dim-4/5 condensate weights that parametrically mimic the stabilizing role those higher-dimension terms would play, as a test of whether a stable Borel window can dynamically form near threshold.

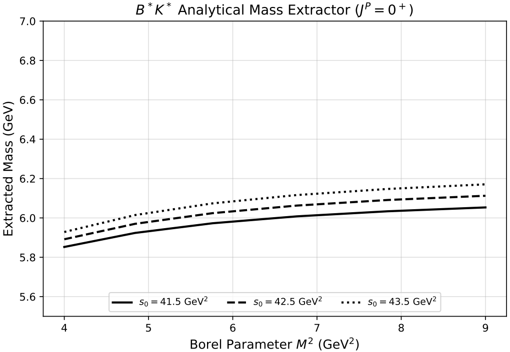
Phase 2: Numerical Mass Extraction – Setup

- Input parameters: $m_b = 4.18$ GeV, $m_s = 0.096$ GeV, chiral limit for u, d; standard condensate values $\langle \bar{q}q \rangle = -(0.24\ \text{GeV})^3$, $\langle G^2 \rangle = 0.012\ \text{GeV}^4$, $\langle \bar{q}Gq \rangle = 0.8\,\langle \bar{q}q \rangle$ (in appropriate units).
- The four analytic OPE pieces from Phase 1 are mapped onto an approximate spectral density $\rho(s) = \rho^{(0)} + \rho^{(3)} + \rho^{(4)} + \rho^{(5)}$ using the threshold phase-space ansatz above.
- Borel moments are then computed numerically:

$$\mathcal{L}_0(M^2, s_0) = \int_{s_{min}}^{s_0} \rho(s)\, e^{-s/M^2}\, ds, \qquad \mathcal{L}_1(M^2, s_0) = \int_{s_{min}}^{s_0} s\, \rho(s)\, e^{-s/M^2}\, ds.$$

- Mass extraction: $m(M^2, s_0) = \sqrt{\mathcal{L}_1/\mathcal{L}_0}$ – exactly the daughter-sum-rule ratio used for the ρ-meson and the QHO ground state.

Phase 2: Borel Stability Plot



Three curves for $s_0 = 41.5,\ 42.5,\ 43.5$ GeV^2.

- All three curves form a slowly-rising plateau around 5.85–6.2 GeV over the Borel window $M^2 \in [4, 10]$ GeV^2.
- Mild residual M^2-dependence signals that the Dim-5 OPE truncation and the phenomenological ansatz are not yet fully converged.

Interpreting the B^*K^* Result

- The Borel-stable mass extracted from this phenomenological model lies in the ≈ 5.9–6.2 GeV region for $s_0 \sim 42$ GeV^2, consistent with (but not yet a precision prediction for) a near-threshold B^*K^* 0^+ molecular state.
- This should be read as a *proof-of-concept / viability test* of the molecular picture under the stated phenomenological ansatz, not as a final, dimension-complete QCDSR prediction.
- **Caveat, stated explicitly:** the threshold phase-space mapping and the effective condensate weights are simplifying choices, made to keep a first analysis tractable; a complete calculation requires the genuine momentum-space spectral density and the true Dim-6 (and higher) condensate contributions.

Form Factors at $Q^2 = 0$

Mode	FF	Hydrogenic		Gaussian	
		Bare	Corrected	Bare	Corrected
$B_c^- \to \eta_c$	f_s	1.820	5.340	1.820	5.370
	f_+	0.197	0.605	0.199	0.610
	f_0	0.194	0.595	0.195	0.598
	f_T	0.301	0.929	0.304	0.936
$B_c^- \to J/\psi$	V	0.423	1.280	0.426	1.290
	A_0	0.269	0.855	0.271	0.861
	A_1	0.266	0.811	0.267	0.815
	A_2	0.265	0.807	0.266	0.811
	T_0	0.820	2.630	0.827	2.660
	T_1	0.327	0.957	0.328	0.961
	T_2	0.107	0.311	0.107	0.313

Table: Form factors at $Q^2 = 0$ for Hydrogenic and Gaussian wave-function models. Corrected values include the non-relativistic Coulomb enhancement.

Partial Decay Widths of B_c^-

Decay Channel	Hydrogenic		Gaussian	
	Bare	Corrected	Bare	Corrected
$B_c^- \to \eta_c \pi^-$	1.950	18.300	1.860	17.500
$B_c^- \to \eta_c K^-$	0.155	1.460	0.148	1.400
$B_c^- \to \eta_c \rho^-$	5.360	50.700	5.130	48.600
$B_c^- \to \eta_c K^{*-}$	0.285	2.700	0.273	2.590
$B_c^- \to \eta_c e^- \bar{\nu}_e$	12.200	118.000	11.100	107.000
$B_c^- \to \eta_c \mu^- \bar{\nu}_\mu$	12.200	117.000	11.000	106.000
$B_c^- \to \eta_c \tau^- \bar{\nu}_\tau$	4.020	39.900	3.390	33.600
$B_c^- \to J/\psi\, \pi^-$	3.480	35.200	3.340	33.800
$B_c^- \to J/\psi\, K^-$	0.274	2.770	0.263	2.660
$B_c^- \to J/\psi\, \rho^-$	11.100	103.000	10.500	97.900
$B_c^- \to J/\psi\, K^{*-}$	0.621	5.810	0.588	5.510
$B_c^- \to J/\psi\, e^- \bar{\nu}_e$	66.700	655.000	59.500	585.000
$B_c^- \to J/\psi\, \mu^- \bar{\nu}_\mu$	66.400	653.000	59.300	583.000
$B_c^- \to J/\psi\, \tau^- \bar{\nu}_\tau$	18.200	183.000	15.000	151.000

Table: Partial decay widths (in units of 10^{-7} eV) for nonleptonic and semileptonic B_c^- decay channels.

Outlook for This Calculation

- **Immediate next step:** extend the OPE to dimension 6 (four-quark condensates under vacuum saturation), the same order at which the ρ-meson and pion sum rules already achieve $\sim$10% accuracy.
- **Medium term:** replace the phase-space ansatz with the full analytic continuation of the x-space correlator (double dispersion relation in two Borel masses, as in the pion form-factor analysis).
- **Cross-check:** compare against a compact-tetraquark current with the same J^P, and against any available lattice or experimental candidate near the B^*K^* threshold.
- Longer term: evaluate genuine Dim-8/Dim-10 vacuum-saturated condensates rather than using effective weights.

Recap: Where Standard QCDSR Struggle

- Two-point function sum rules are built for *static* quantities (masses, couplings) using a *local* OPE around $x = 0$.
- Problems arise when:
  - the relevant physics is genuinely *nonlocal* (form factors at large momentum transfer, exclusive processes);
  - the hadron wavefunction/structure itself is the object of interest;
  - multiple, well-separated energy scales appear simultaneously.
- Several extensions of the original SVZ method were developed precisely to address these limitations.

Light-Cone QCD Sum Rules (LCSR)

- Instead of expanding the correlator in *local* vacuum condensates, expand in light-cone distance, organizing the OPE by *twist* rather than by operator dimension alone.
- The nonperturbative input becomes the hadron light-cone distribution amplitude $\phi(x, \mu)$ – a generalization of the parton-momentum-fraction wavefunction.
- Particularly suited to exclusive processes at moderately large momentum transfer: form factors, transition amplitudes, and many heavy-hadron decay matrix elements.
- Conceptually: replaces the vacuum-condensate expansion with a hadron-state matrix-element expansion, while keeping the same Borel/duality machinery.

Other Extensions of the SVZ Method

- **HQET sum rules** – expand directly in $1/m_Q$ for heavy-quark systems, removing the heavy mass from the start (as used implicitly in the B^*K^* analysis above).
- **Finite-temperature / finite-density sum rules** – condensates become medium-dependent, used to study in-medium hadron properties.
- **Three-point function sum rules** – access couplings and transition form factors directly (used above for $F_\pi(Q^2)$).
- **Double/multiple Borel sum rules** – needed whenever more than one large momentum scale is present (e.g. semileptonic form factors, the pion form factor itself).
- **QCD sum rules for multiquark/exotic currents** – the framework used in the B^*K^* example, now a very active subfield given the ongoing discovery of exotic states.

Summary

- QCD sum rules connect the QCD Lagrangian to hadron observables via OPE + dispersion relations + Borel transform + quark-hadron duality.
- Every essential idea is already present in the exactly-solvable quantum harmonic oscillator, which also calibrates the expected accuracy: $\sim$ 10–20%.
- Standard QCDSR remain in active use today precisely because of their flexibility and speed, especially for exotic hadron spectroscopy where lattice QCD and experiment are still catching up.
- We illustrated this with a B^*K^* ($J^P = 0^+$) hadronic-molecule calculation, built on an automated OPE plus an explicitly-justified phenomenological ansatz.
- Extensions such as light-cone sum rules, HQET sum rules, and multiquark-current sum rules push the method into regimes the original SVZ formulation could not reach.

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- The original QCD sum rule method is due to M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov (1979).

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Thank you!

Questions & discussion