# Wavelet-Based QFT Formulation

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# Outline of the talk:

- *Motivation*
- *Background*
- *Introduction to Daubechies Wavelet Basis*
- *Free Scalar Field Theory in Wavelet Basis*
- *Conclusion and Outlook*

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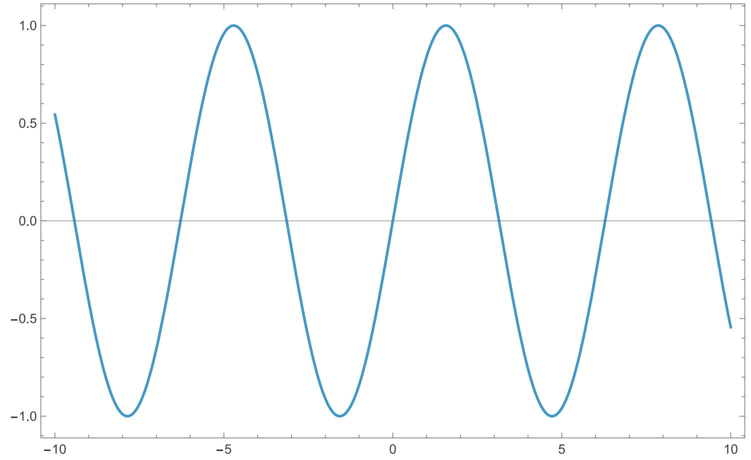
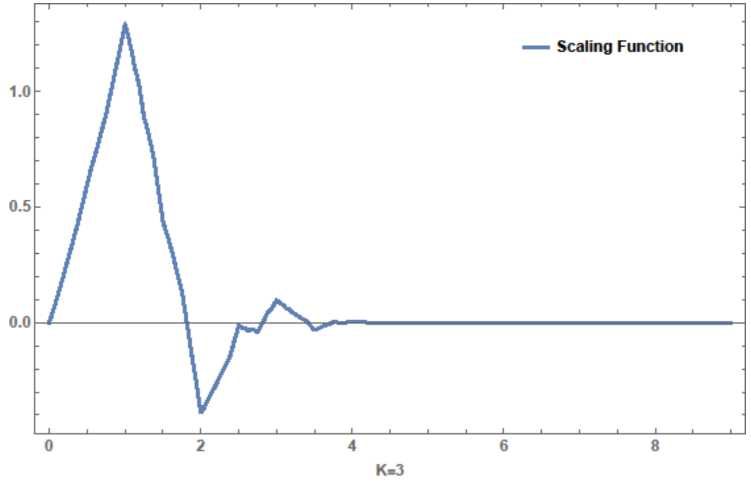
# Motivation: Why non-perturbative Formalism?

- **The Failure of the perturbation theory**
  - *The perturbation theory fails, when the coupling is strong (e.g., Low-energy QCD).*
  - *Important phenomena of physics are invisible to a finite order in perturbation theory (Tunnelling)*
- **The Bound state problem:**
  - *Perturbation theory describes scattering processes but generally cannot describe bound states that are dynamically generated by strong interactions, such as mesons and baryons in QCD.*
- **Real Time Dynamics:**
  - *Perturbation theory is often formulated in Euclidean time (imaginary time). The real-time evaluation of the wavefunction needs the non-perturbative analysis.*

# Motivation: Why Hamiltonian Formulation in Daubechies Wavelet Basis?

- **The Problem with Standard Bases (Fourier/Momentum)**
  - *Plane waves ($e^{ikx}$) are delocalized all over the space.*
  - *They are inefficient at describing non-perturbative structure (like kinks, bound states).*
- **The Wavelet Solution:**
  - ***Localization:*** *Daubechies wavelets have* **compact support** *(they are non-zero only in a small region), making them ideal for capturing local physics and reduce the infinite interaction terms to finite, calculable values.*
  - ***Renormalization:*** *Wavelets separate physics by length scale (Multi-Resolution Analysis).*



# Background

- **Wilson's Pioneering Work:**
  - *The use of wavelets for nonperturbative analysis of quantum field theories (QFTs) first emerged in a* ***1965*** *work by K.G Wilson titled "****Model Hamiltonians for Local Quantum Field Theory****".*[KG Wilson, Phys. Rev. **140**, B445, (1965)]
  - *He wanted to construct a set of functions, which he called "Wave-Packet" functions, that rapidly goes to zero as one move away from the assigned region in both position and momentum space.*
  - *But he claimed to have constructed a function which is either fully localized in position and have a long tail in the momentum space and vice-versa.*

- **The work of Daubechies:**
  - *Nearly three decade later* ***1988*** *Daubechies developed a orthonormal bases which has a strikingly similar properties to the Wilson's* ***wave-packet*** *functions.*[I. Daubechies, Comm. Pure Appl. Math. 41, 909 (1988)]
  - *With this discovery in* ***1994*** *Wilson emphasized the importance of wavelets for non-perturbative analysis of quantum chromodynamics (QCD) in 1994.*[KG Wilson et al. ,Phys. Rev. D **49**, 6720,(1994)]

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# Introduction to Daubechies wavelet basis:
## Origins and Key Features

- **Origins and Key Features:**

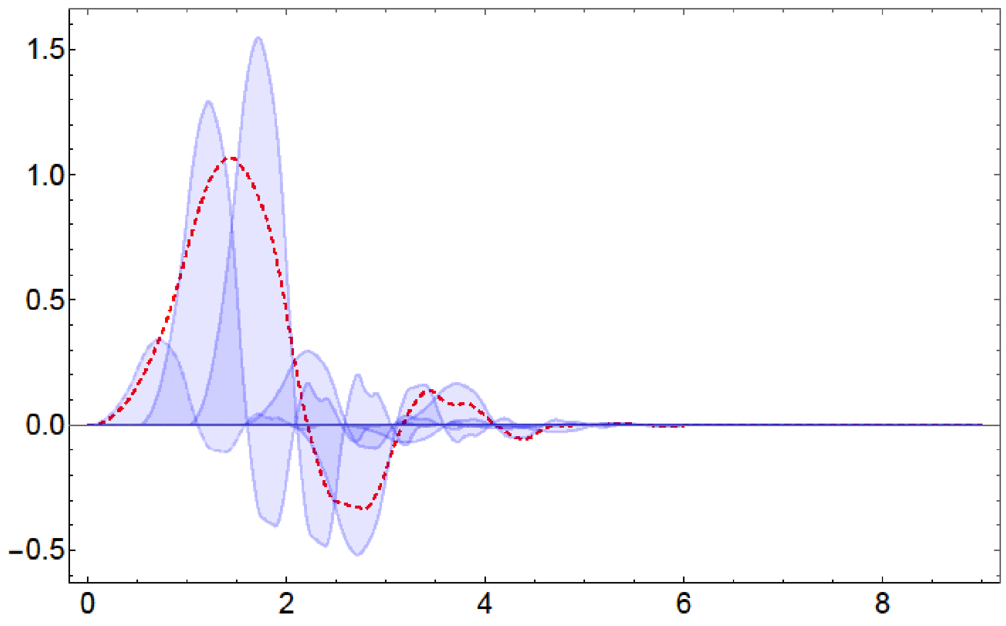
  - *This wavelet was introduced by **Ingrid Daubechies** in the year **1988**.*

  - *This wavelet basis functions possesses the **compact support** feature.*

  - *The basis functions can be generated from a single basis function known as the **mother scaling function**, defined by the refinement following equation,*

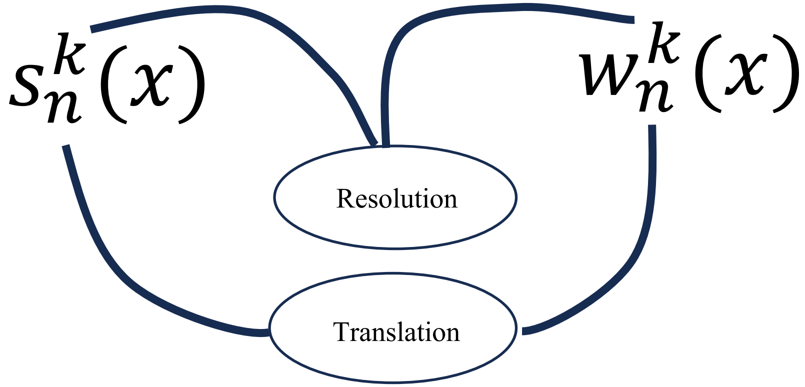
$$s(x) = \sum_{n=0}^{2K-1} DT^n s(x)\,.$$

  - *All the basis functions are orthonormal to each other.*

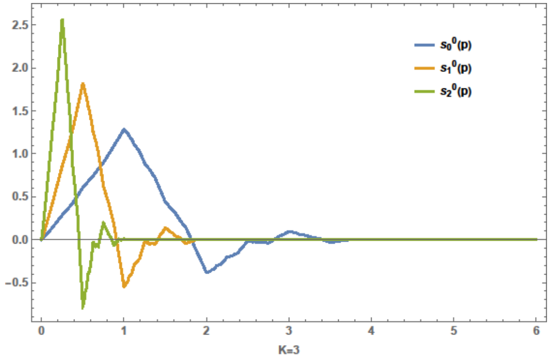


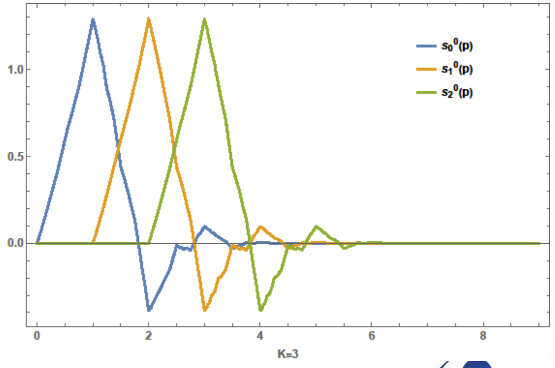
# Introduction to Daubechies wavelet basis

- **The Two Components:**
  - *The basis is constructed from two fundamental functions:* ***The scaling function*** $(\boldsymbol{s_m^k(x)})$ *and* ***The wavelet Function*** $(\boldsymbol{w_m^k(x)})$***.*** *The scaling function is called the* ***Approximate (coarser) component.*** *The wavelet function is called the* ***detailed component.***




- **Characterization by Indices:**
  - ***Resolution Index*** $(\boldsymbol{k})$: *Increasing k squeezes the function to* $\frac{1}{2}$ *of its original width (increasing momentum), while keeping the norm fixed.*
  - ***Translation Index*** $(\boldsymbol{n})$: *Changing n translates the localized function strictly by one spatial unit.*



# Introduction to Daubechies wavelet basis

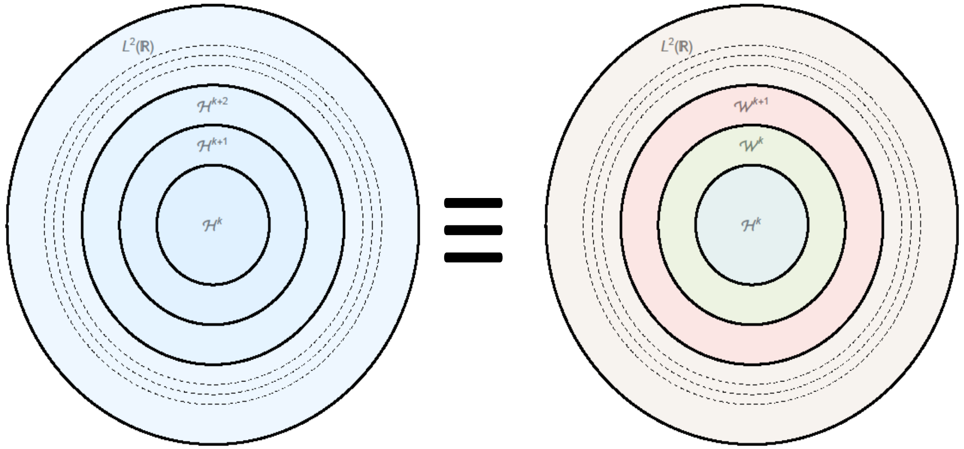
- **Implementing QFT Cutoffs:**
  - *To compute the Hamiltonian, we truncate the infinite basis by limiting both indices.*
  - ***Ultraviolet (UV) Cutoff:*** *Implemented by restricting the maximum resolution index* $(k)$.
  - ***Infrared (IR) Truncation:*** *Achieved by limiting the translation index* $(n)$, *which restricts the physical volume.*

- **Constructing the Hilbert Space:**
  - *The entire Hilbert space can be formed with* ***only the scaling function*** *or the combination of* ***scaling and wavelet functions****.*
  - ***Multi-Resolution Analysis (MRA):*** *The space at resolution k is the direct sum of the lower-resolution space and the added wavelet details:*

$$\mathcal{L}^2(\mathbb{R}) = \lim_{k\to\infty} \mathcal{H}^k$$

*By recursive application of this equation, the space of square integrable functions can be constructed,*

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{H}^k \oplus \mathcal{W}^k \oplus \mathcal{W}^{k+1} \oplus \mathcal{W}^{k+2} \ldots$$



# Free Scalar Field Theory in wavelet basis

- Free Scalar field theory Hamiltonian:

$$\mathbb{H}_0 = \int dx \left( \frac{1}{2} [\pi^2 + (\partial_x \phi)^2 + m^2 \phi^2] \right).$$

- Field operators in terms of the scaling functions:

$$\phi(x,t) = \sum_{n=-\infty}^{\infty} \phi_n^{s,k}(t) s_n^k(x) + \sum_{\substack{n=-\infty \\ k \geq l}}^{\infty} \phi_n^{w,l}(t) w_n^l(x)$$

$$\pi(x,t) = \sum_{n=-\infty}^{\infty} \pi_n^{s,k}(t) s_n^k(x) + \sum_{\substack{n=-\infty \\ k \geq l}}^{\infty} \pi_n^{w,l}(t) w_n^l(x),$$

- Free scalar theory Hamiltonian in Daubechies Wavelet basis:

$$\mathbb{H} = \mathbb{H}_{ss} + \mathbb{H}_{ww} + \mathbb{H}_{sw}$$

$$\mathbb{H}_{ss} = \frac{1}{2} \left( \sum_n \pi_n^{s,k}(t) \pi_n^{s,k}(t) + \sum_{m,n} \mathcal{D}^k_{ss,mn} \phi_m^{s,k}(t) \phi_n^{s,k}(t) + \sum_n \mu^2 \phi_n^{s,k}(t) \phi_n^{s,k}(t) \right)$$

$$\mathbb{H}_{ww} = \frac{1}{2} \left( \sum_{n,l \geq k} \pi_n^{w,l}(t) \pi_n^{w,l}(t) + \sum_{\substack{m,n \\ l \geq k}} \mathcal{D}^{kl}_{ww,mn} \phi_m^{w,k}(t) \phi_n^{w,l}(t) + \sum_{\substack{n \\ l \geq k}} \mu^2 \phi_n^{w,l}(t) \phi_n^{w,l}(t) \right)$$

$$\mathbb{H}_{sw} = \sum_{\substack{m,n \\ l \geq k}} \mathcal{D}^{kl}_{sw,mn} \phi_m^{s,k}(t) \phi_n^{w,l}(t)$$

Where,

$$\mathcal{D}^k_{ss,mn} = \int \frac{\partial s_m^k(x)}{\partial x} \frac{\partial s_n^k(x)}{\partial x} dx,$$

$$\mathcal{D}^{kl}_{ww,mn} = \int \frac{\partial w_m^k(x)}{\partial x} \frac{\partial w_n^l(x)}{\partial x} dx\,,$$

$$\mathcal{D}^{kl}_{sw,mn} = \int \frac{\partial s_m^k(x)}{\partial x} \frac{\partial w_n^l(x)}{\partial x} dx$$

These overlap integrals represents the interaction between different modes

So, the scalar field theory Hamiltonian can be written as,

$$\mathbb{H} = \Pi^T \Pi + \frac{1}{2} \Phi^T \Omega^2 \Phi$$

Where,

$$\Omega^2 = \begin{pmatrix} \mathcal{D}^k_{ss} & \mathcal{D}^{kl}_{sw} \\ \mathcal{D}^{lk}_{ws} & \mathcal{D}^{lm}_{ww} \end{pmatrix}$$

$$H = \begin{array}{|c|c|} \hline H_{ss} & H_{sw} \\ \hline H_{ws} & H_{ww} \\ \hline \end{array}$$

# Free Scalar Field Theory in wavelet basis

- **First attempt to quantize the theory:**

  - The first attempt to quantize the theory was done by **Polyzou et al** [F. Bulut and W. N. Polyzou, Phys. Rev. D 87, 116011 (2013)], by defying the creation and annihilation operator as the following,

$$a_n^{s,k}(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\gamma}\phi_n^{s,k}(t) + i\frac{1}{\sqrt{\gamma}}\pi_n^{s,k}(t)\right),$$

$$a_n^{s,k\dagger}(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\gamma}\phi_n^{s,k}(t) - i\frac{1}{\sqrt{\gamma}}\pi_n^{s,k}(t)\right).$$

by assuming this annihilation operator will annihilate the vacuum,

$$a_n^{s,k}|0\rangle = 0$$

we can derive the following expression of $\gamma$.

$$\gamma = \frac{1 \pm \sqrt{1 - 4\langle 0|\phi_n^{s,k}(t)\phi_n^{s,k}(t)|0\rangle\langle 0|\pi_n^{s,k}(t)\pi_n^{s,k}(t)|0\rangle}}{2\,\langle 0|\phi_m^{s,k}(t)\phi_n^{s,k}(t)|0\rangle},$$

The inverse of these operators can be written as,

$$\phi_n^{s,k}(t) = \frac{1}{\sqrt{2\gamma}}\left(a_n^{s,k}(t) + a_n^{s,k\dagger}(t)\right)$$

$$\pi_n^{s,k}(t) = -i\sqrt{\frac{\gamma}{2}}\left(a_n^{s,k}(t) - a_n^{s,k\dagger}(t)\right)$$

As $\phi_n^{s,k}(t)$ is real

$$\phi_n^{s,k^*}(t) = \phi_n^{s,k}(t)$$

Hence, $\gamma$ is supposed to be a **real quantity**.

But from Cauchy-Schwarz inequality, we can prove that,

$$\langle 0|\phi_n^{s,k}(t)\phi_n^{s,k}(t)|0\rangle\langle 0|\pi_n^{s,k}(t)\pi_n^{s,k}(t)|0\rangle \geq \frac{1}{4}$$

So, $\gamma$ comes out to be a **complex quantity**.

So what went wrong ???

From the Bogoliubov Transformation, we can prove that the annihilation operators can be written as a linear combination of the momentum creation and annihilation operators,

$$a_n^{s,k}(t) = \int \left(a^\dagger(p) + a(p)\right) dp$$

So, this definition of annihilation operator will not annihilate the real vacuum,

$$a_n^{s,k}(t)|0\rangle = \int \left(a^\dagger(p) + a(p)\right) dp\ |0\rangle$$
$$= \int dp\ a^\dagger(p)|0\rangle$$

$$a_n^{s,k}(t)|0\rangle = 0$$

# Free Scalar Field Theory in wavelet basis

- **Our definition of the Creation and annihilation operator:**
  - The compact form of the Hamiltonian in wavelet basis,

$$\mathbb{H} = \frac{1}{2}(\Pi^T\Pi + \Phi^T\Omega^2\Phi).$$

  - The new set of creation and annihilation operator in this basis is defined as,

$$a(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\Omega}\Phi(t) + i\frac{1}{\sqrt{\Omega}}\Pi(t)\right),$$

$$a^\dagger(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\Omega}\Phi(t) - i\frac{1}{\sqrt{\Omega}}\Pi(t)\right).$$

Whether this newly defined annihilation operator can annihilate the vacuum

?

From Bogoliubov transformation, it can be proved that,

$$a_m(t) = \int a(p)\, dp\,,$$

Hence,

$$a_m(t)|0\rangle = 0.$$

The Hamiltonian in terms of these new creation and annihilation operator takes the following form,

$$\mathbb{H} = \frac{1}{2}[(a)^T\Omega\, a + Tr[\Omega]\,].$$

$$\mathbb{H} = \frac{1}{2}\left[a^{s,k^T}\Omega^{kk}_{ss}a^{s,k} + a^{s,k^T}\Omega^{kl'}_{sw}a^{w,l'} + a^{w,l^T}\Omega^{lk}_{ws}a^{s,k} + a^{w,l^T}\Omega^{ll'}_{ww}a^{w,l'}\right] + Tr[\Omega]$$

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# Free Scalar Field Theory in wavelet basis:

- **The Fock Space basis:**
  - *A generic infinite particle state is defined by,*

$$\left|m^{s}_{-\infty}, \dots, m^{s}_{p}, \dots, m^{s}_{\infty} : m^{w}_{-\infty}, \dots, m^{w}_{p}, \dots, m^{w}_{\infty}\right\rangle$$

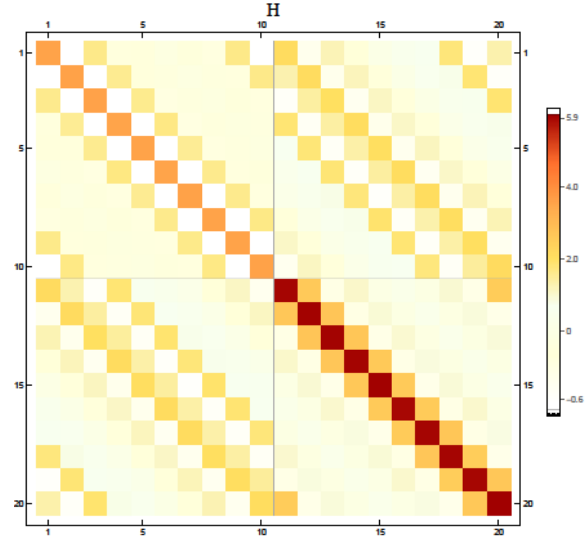
  - *The action of the creation and annihilation operator in this state is given by,*

$$a_p^{s,k\dagger}\left|m^{s,k}_{-\infty}, \dots, m^{s,k}_{p}, \dots, m^{s,k}_{\infty}\right\rangle = \sqrt{m^{s,k}_{p} + 1}\left|m^{s,k}_{-\infty}, \dots, (m+1)^{s,k}_{p}, \dots, m^{s,k}_{\infty}\right\rangle$$

$$a_p^{s,k}\left|m^{s,k}_{-\infty}, \dots, m^{s,k}_{p}, \dots, m^{s,k}_{\infty}\right\rangle = \sqrt{m^{s,k}_{p}}\left|m^{s,k}_{-\infty}, \dots, (m-1)^{s,k}_{p}, \dots, m^{s,k}_{\infty}\right\rangle$$

*Within the truncated resolution-1 subspace, the Hamiltonian for the single-particle Fock space basis of a simulation box defined by $0 \leq L \leq 10$ is given by the matrix plot, (We applied the periodic boundary condition),*

$$\mathrm{H} = \begin{bmatrix} \Omega_{ss} & \Omega_{sw} \\ \Omega_{ws} & \Omega_{ww} \end{bmatrix}$$

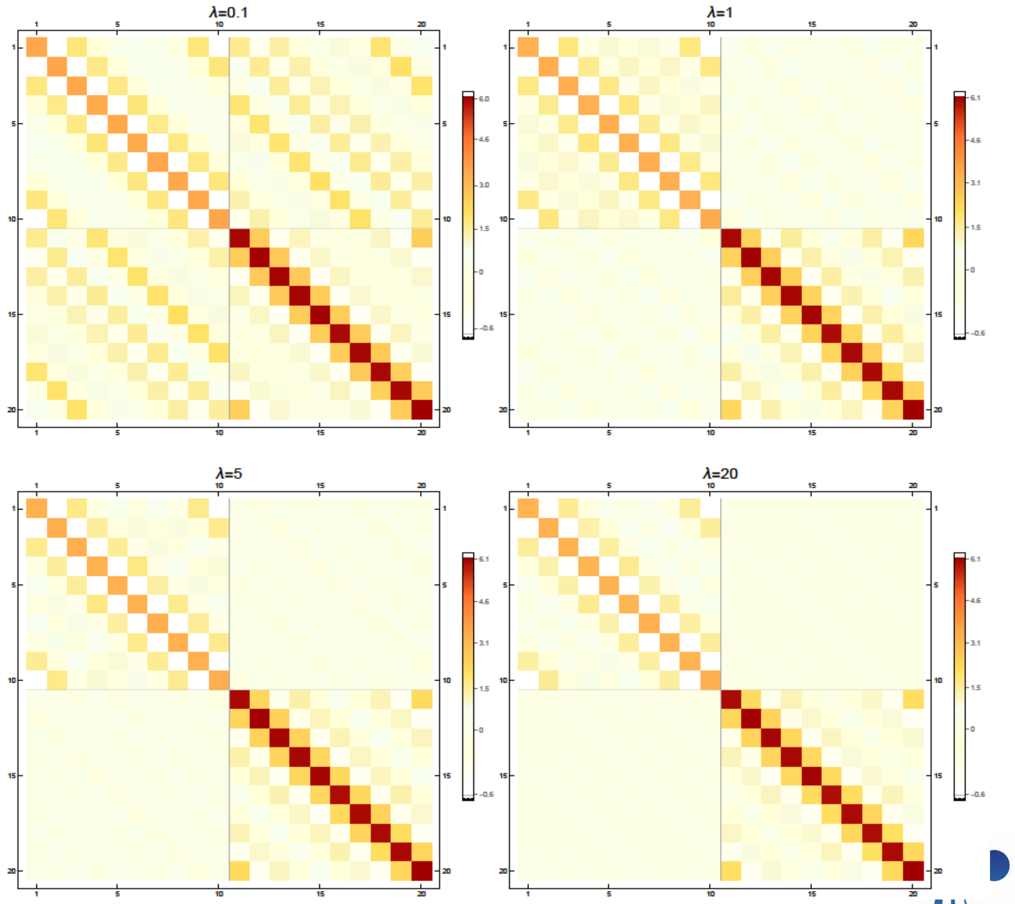



*The dimension of the Hamiltonian matrix is=20*20*

# Free Scalar Field Theory in wavelet basis:

*The one of the many methods to decouple the approximate and the detailed degrees of freedom is the* ***similarity renormalization group (SRG)*** *approach which involves solving the following differential equation,*

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)]$$

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# Free Scalar Field Theory in wavelet basis:

| $States$ | $Analytical$ | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
|---|---|---|---|---|---|---|
| $0p$ | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| $1p\ 0m$ | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| $2p\ 0m$ | 2.000000 | 2.000000 | 2.000000 | 2.000000 | 2.000000 | 2.000000 |
| $2p\ \pm 2m$ | 2.362020 | 2.368770 | 2.362481 | 2.362049 | 2.362021 | 2.362020 |
| $3p\ 0m$ | 3.000000 | 3.000000 | 3.000000 | 3.000000 | 3.000000 | 3.000000 |
| $2p \pm 4m$ | 3.211938 | 3.427144 | 3.231760 | 3.213295 | 3.212025 | 3.211944 |
| $3p\ 0m \pm 2m$ | 3.362020 | 3.368770 | 3.362481 | 3.362049 | 3.362021 | 3.362020 |
| $3p\ 4m \pm 2m$ | 3.967989 | 4.082342 | 3.978361 | 3.968697 | 3.968034 | 3.967992 |
| $4p\ 0m$ | 4.000000 | 4.000000 | 4.000000 | 4.000000 | 4.000000 | 4.000000 |

# Summary and Outlook:

- **Summary of Our Formulation:**
  - We have successfully developed a localized, non-perturbative Hamiltonian formulation using Daubechies wavelets.
  - By employing the similarity renormalization group (SRG), we obtain an effective Hamiltonian that can be studied at significantly reduced computational cost. We demonstrate the method for the case of the 1+1-dimensional free scalar field theory
  - The numerical results demonstrate that the low-lying eigenvalues systematically approach to their analytical values with increasing resolution, confirming the consistency of the method.

- **The Classical Bottleneck:**
  - While this discrete formulation works beautifully, extending it to higher spatial dimensions or drastically increasing the resolution ($k$) leads to an exponential explosion in the size of the Hilbert space.
  - Exact classical diagonalization eventually hits a hard computational wall.

- **The Only Way Forward: Qubitization:**
  - The natural and necessary next step is the **Qubitization** of these wavelet-based Hamiltonians—mapping our discrete bosonic operators to qubits to perform exact real-time quantum simulations of strongly interacting QFTs.

# References:


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# Thank You