Light-Cone Distribution Amplitudes: Why, What and How

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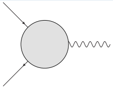
Form Factors

- In nonrelativistic quantum mechanics

$$F(\vec{q}^{\,2}) = \int d^3r\, e^{i\vec{q}\cdot\vec{r}} |\Psi(\vec{r})|^2 = \int d^3q'\, \Psi(q')\Psi^*(q-q')$$



- In quantum field theory

  — electromagnetic nucleon

$$\langle N(p+q)|j_\mu^{\rm em}|N(p)\rangle = \bar{u}(p+q)\Big[\gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_N}F_2(Q^2)\Big]u(p)\,, \qquad Q^2 = -q^2$$

  — electromagnetic pion

$$\langle \pi(p')|j_\mu^{\rm em}|\pi(p)\rangle = (p+p')_\mu F(Q^2)$$

  — pion–photon transition

$$\int d^4y\, e^{iq_1y}\langle \pi^0(p)|T\{j_\mu^{\rm em} j_\nu^{\rm em}\}|0\rangle = i\epsilon_{\mu\nu\alpha\beta}q_1^\mu q_2^\nu F(q_1^2,q_2^2)\,, \qquad q_2 = q_1 + p$$

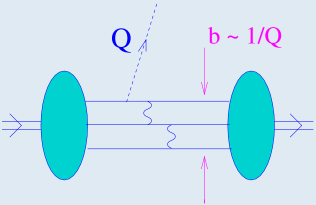
  — weak decays

$$B \to \gamma\ell\nu_\ell$$
$$B \to \pi\ell\nu_\ell\,, \qquad B \to K^*\ell\nu_\ell\ \ldots$$
$$\Lambda_b \to p\ell\nu_\ell$$
$$\ldots$$

- Extensive experimental studies

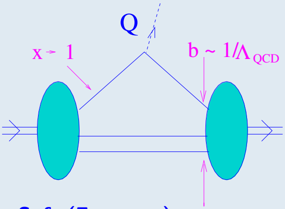
How to transfer a large momentum to a loosely bound system?

Two possible mechanisms:



Hard rescattering:

Small b
Average $0 < x < 1$



Soft (Feynman):

Average b
Large $x \to 1$

- Hard rescattering dominates for simplest reactions. In these cases, soft contributions are suppressed by a power $1/Q^2$, same as hard contributions coming e.g. from Fock states with extra gluons
- Hard contributions involve a part of the wave function with the smallest amount of partons and at small transverse separations — the Light-Cone Distribution Amplitude
- Soft contributions involve a part of the wave function with one quark carries most of the momentum and any amount of partons with almost zero momenta

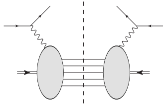
Interpretation

- Hadrons in infinite momentum frame can be described in terms of multicomponent wave functions — Fock states

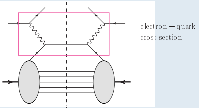
$$|\text{Meson}\rangle = \begin{pmatrix} |\bar{q}q\rangle \\ |\bar{q}gq\rangle \\ |\bar{q}q\bar{q}q\rangle \\ \ldots \end{pmatrix} = \begin{pmatrix} \Psi_2(x, k^\perp) \\ \Psi_3(x_i, k_i^\perp) \\ \Psi_4(x_i, k_i^\perp) \\ \ldots \end{pmatrix} \qquad |\text{Baryon}\rangle = \begin{pmatrix} |qqq\rangle \\ |qqqg\rangle \\ |qqq\bar{q}q\rangle \\ \ldots \end{pmatrix}$$

Variables: $x_i \leftarrow$ momentum fractions; $k_i^\perp \leftarrow$ transverse momenta

- Inclusive processes (e.g. Deep Inelastic Scattering)



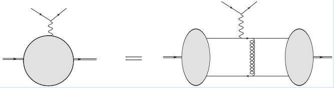
$= \sum_N$



$$\text{Parton Distribution Function (PDF)} = \sum_N \int dk_i^\perp \int dx_i |\Psi_N(x_i, k_i^\perp)|^2 \delta(x_1 - x)$$

$\Leftarrow$ Probability to find a quark with given momentum fraction (one-particle density)

- Exclusive processes (e.g. Form Factors)



$$\text{Distribution Amplitude (LCDA)} = \int dk_\perp \Psi_2(x, k_\perp)$$

$\Leftarrow$ Wave function of the lowest Fock state (valence quarks only) at zero transverse separation

Definition in Quantum Field Theory

- LCDAs $\stackrel{!}{=}$ vacuum-to-hadron matrix elements of light-ray operators

  Pion LCDA

$$\langle 0|\bar{u}(tn)[tn,0]\not{n}d(0)|\pi^+(p)\rangle = if_\pi(p\cdot n)\int_0^1 dx\, e^{it(p\cdot n)x}\phi_\pi(x,\mu)$$

here $t \in \mathbb{R}$, $n^2 = 0$ (a light-like four-vector), f_π is the pion decay constant, $[tn, 0]$ is the Wilson line

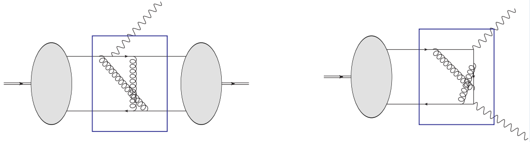
$$[t_1 n, t_2 n] = \mathrm{Pexp}\left\{ig\int_{t_2}^{t_1} d\alpha\, n_\mu A^\mu(\alpha t_1 + (1-\alpha)t_2)\right\}$$

Nucleon (proton) LCDA

$$\langle 0|u(t_1 n)[t_1 n,0]u(t_2 n)[t_2 n,0]d(t_3 n)[t_3 n,0]|P(p)\rangle = \ldots$$

QCD factorization

- Hard rescattering contributions can be factorized



$$F_\pi^{\rm em}(Q^2) = \frac{1}{Q^2}\int_0^1 dx \int_0^1 dy\, \underbrace{\phi_\pi(x,\mu)}_{\rm LCDA}\, H\Big(x,y,\ln Q^2/\mu^2,\alpha_s(\mu)\Big)\, \underbrace{\phi_\pi(y,\mu)}_{\rm LCDA}$$

$$F_{\pi\gamma^*\gamma}(Q^2) = \frac{1}{Q^2}\int_0^1 dx\, \widetilde{H}\Big(x,\ln Q^2/\mu^2,\alpha_s(\mu)\Big)\, \underbrace{\phi_\pi(y,\mu)}_{\rm LCDA}, \qquad Q^2=-q_1^2\,,\ q_2^2=0$$

- The LCDAs are universal nonperturbative functions, depend on factorization scale
  Hard functions are process-specific, can be calculated in perturbation theory
- In B-decays, large b-quark mass playes the role of the large scale $Q \sim m_b$
- Soft contributions cannot be factorized in terms of LCDAs $\longrightarrow$ complicated

Leading order results and scale dependence

$$F_\pi^{\rm em}(Q^2) = \frac{8\pi\alpha_s f_\pi^2}{9Q^2}\left|\int_0^1 dx\,\frac{\phi_\pi(x)}{1-x}\right|^2 \qquad\qquad F_{\pi\gamma^*\gamma}(Q^2) = \frac{\sqrt{2}f_\pi}{3Q^2}\int_0^1 dx\,\frac{\phi_\pi(x)}{1-x}$$

- The ERBL equation

$$\mu^2\frac{d}{\mu^2}\phi_\pi(x,\mu) = \frac{\alpha_s(\mu)}{2\pi}\int_0^1 dy\,V(x,y)\phi_\pi(y,\mu)$$

$$V(x,y) = C_F\left[\frac{1-x}{1-y}\left(1+\frac{1}{x-y}\right)\theta(x-y)+\frac{x}{y}\left(1+\frac{1}{y-x}\right)\theta(y-x)\right]_+$$

is solved by an expansion in orthogonal Gegenbauer polynomials

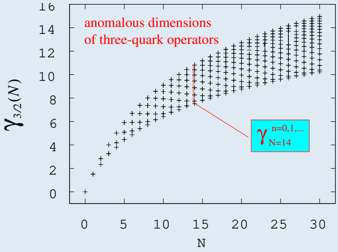
$$\phi_\pi(x,\mu) = \underbrace{6x(1-x)}_{\text{"asymptotic LCDA"}}\left[1+\sum_{n=2}^{\infty}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_n/2\beta_0}\underbrace{a_n(\mu_0)}_{\text{parameters}}C_n^{3/2}(2x-1)\right]$$

- The most important nonperturbative parameter — the inverse moment

$$\boxed{\int_0^1 dx\,\frac{\phi_\pi(x,\mu)}{1-x} = 3\Big[1+a_2(\mu)+a_4(\mu)+a_6(\mu)+\dots\Big]}$$

- Baryon LCDAs

$$\phi_N(x_1,x_2,x_3,\mu) = \underbrace{120x_1x_2x_3}_{\text{"asymptotic LCDA"}} \left[1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_{nk}/2\beta_0} \underbrace{a_{nk}(\mu_0)}_{\text{parameters}}\underbrace{P_{nk}(x_1,x_2,x_3)}_{\text{orthogonal polynomials}}\right]$$

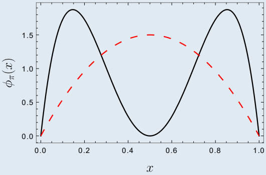


- Only the Dirac form factor $F_1(Q^2)$ is factorizable in terms of the LCDAs; leading-order expressions already complicated

How can we learn about the LCDAs?

- Part I: Prehistoric period

- Chernyak–Zhitnitsky Model, 1982
  - — QCD sum rules: Pion LCDA is broader that the asymptotic one
  - — The model allows to describe the existing data
  - — Pion is “special”, its LCDA can be unlike other mesons



— Remains to be a standard reference model until at least 2010 despite heavy critisism

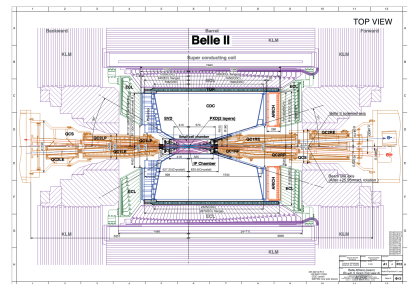
- — QCD sum rules of this type are not reliable
- — Light-cone sum rules $\to \phi_\pi(x=1/2) \simeq 1.2$
- — CZ model implicates very large soft contributions which destroy agreement with data

— Discredited by modern lattice calculations

- Similar models exist for baryons; they are discredited as well

How can we learn about the LCDAs?

- Part II: Phenomenology

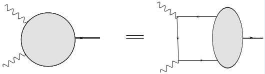


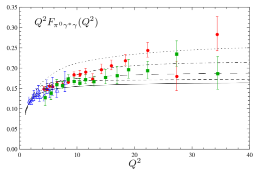
— Belle II detector, SuperKEKB

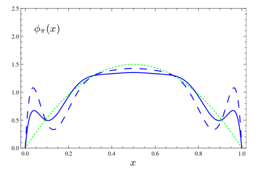
How can we learn about the LCDAs?

- “Gold-plated” process: Pion transition form factor $\gamma^*\gamma \to \pi$

  — highest theory precision, can be measured with good accuracy







from: A. Agaev *et al.* 1205.3968

- Problem: Experimental data from BaBar (2009) and Belle (2012) do not agree; wait for Belle II

  — observed scaling violation indicates possible enhancement close to end points

How can we learn about the LCDAs?

- Part III: Lattice QCD

— from: Forschungszentrum Jülich Photo Gallery

- Methods:

$\mathbb{A}$ Matrix elements of local operators
$\mathbb{B}$ Euclidean correlation functions
$\mathbb{C}$ Matrix elements of nonlocal operators (LaMET)

$\mathbb{A}$: Matrix elements of local operators → coefficients in Gegenbauer expansion

$$\langle 0|\bar u(-n)[n,0]\not n d(n)|\pi^+(p)\rangle = i f_\pi (p\cdot n)\int_0^1 dx\, e^{i(2x-1)(p\cdot n)}\phi_\pi(x,\mu)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_{k=0}^\infty \frac{1}{k!}\langle 0|\bar u(0)(\overleftrightarrow{D}\cdot n)^k \not n d(0)|\pi^+(p)\rangle = f_\pi \sum_{k=0}^\infty \frac{i^{k+1}}{k!}(p\cdot n)^{k+1}\underbrace{\int_0^1 dx\,(2x-1)^k\phi_\pi(x,\mu)}_{\langle\xi^k\rangle}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\langle 0|\bar u(0)\overleftrightarrow{D}_{\{\mu_1}\ldots\overleftrightarrow{D}_{\mu_k}\gamma_{\mu_{k+1}\}}d(0)|\pi^+(p)\rangle}_{\text{lattice}} = i^{k+1}p_{\{\mu_1}\ldots p_{\mu_{k+1}\}}\langle\xi^k\rangle$$

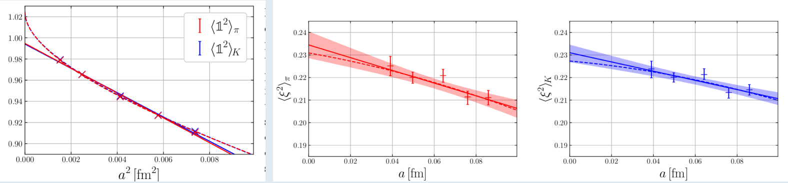
where from Gegenbauer coefficients are obtained by linear relations

$$\langle\xi^0\rangle = 1$$
$$\langle\xi^2\rangle = \frac{1}{5}+\frac{12}{35}a_2$$
$$\langle\xi^4\rangle = \frac{3}{35}+\frac{8}{35}a_2+\frac{8}{77}a_4 \qquad \ldots$$

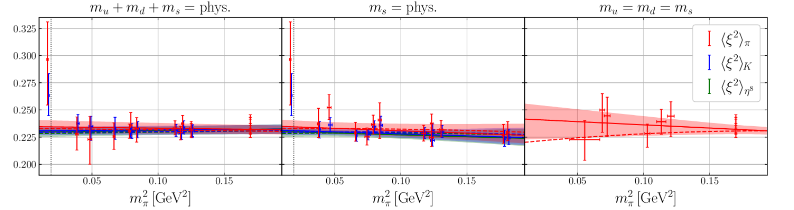
- Advantage: Currently the only method that allows for a full control over statistical and systematic errors
- Disadvantage: Only the first few coefficients accessible; more with gradient flow (?)

Current state-of-the-art: RQCD collaboration, G. Bali *et al.* 1903.12590

- Continuum extrapolation



- Chiral extrapolation



- Nonperturbative Renormalization in RI'/SMOM scheme; three-loop matching to $\overline{\text{MS}}$

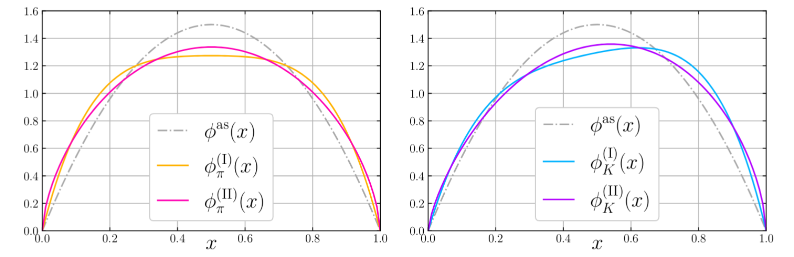
Current state-of-the-art:

RQCD collaboration, G. Bali *et al.* 1903.12590

- At the scale 2 GeV

$$\langle \xi^2 \rangle^\pi = 0.234^{+6}_{-6}(4)_r(4)_a(2)_m \qquad a_2^\pi = 0.101^{+17}_{-17}(12)_r(10)_a(5)_m$$

$$\langle \xi^2 \rangle^K = 0.231^{+4}_{-4}(4)_r(4)_a(1)_m \qquad a_2^K = 0.090^{+10}_{-12}(11)_r(11)_a(4)_m$$



$\mathbb{B}$: Custom-made euclidean (equal time) correlation functions

Method: VB, D. Mueller, 0709.1348
A proof of concept: RQCD collaboration, 1807.06671

- Correlation functions of two local currents are factorizable in terms of LCDAs

$$\langle 0|T\{O_1(\tfrac{z}{2})O_2(-\tfrac{z}{2})\}|\pi(p)\rangle = \frac{if_\pi}{16\pi^2 z^4}\dots\left\{\int_0^1 dx\, e^{i(x-1/2)pz}[1+O(\alpha_s)]\phi_\pi(x) + \mathcal{O}(z^2\Lambda^2_{\rm QCD})\right\}$$

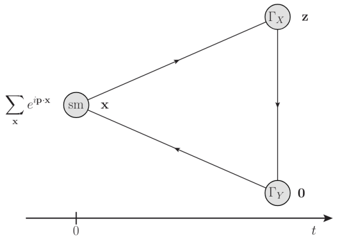
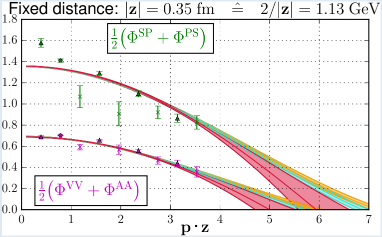
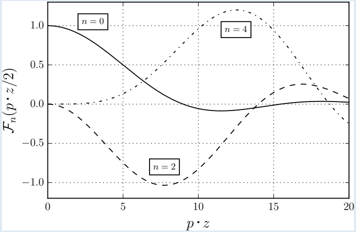


FIG. 4. The relevant triangle diagram, with a smeared interpolating current for the pion at $t = 0$. The Fourier transform corresponds to an incoming pion with momentum $\mathbf{p}$ (for $t > 0$) or an outgoing pion with momentum $-\mathbf{p}$ (if $t < 0$).

! Factorization holds up to power corrections; their contribution is part of systematic uncertainties position space LCDA:

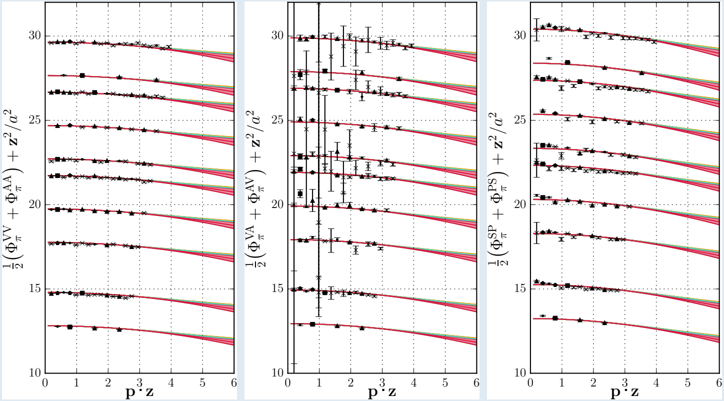
$$\int_0^1 dx\, e^{i(x-1/2)pz}\phi_\pi(x,\mu) = \sum_{n=0}^{\infty} a_n^\pi(\mu)\mathcal{F}_n(pz/2)$$





- Need a large lever arm in (pz) retaining $|z|$ small $\Rightarrow$ large pion momentum
- Can exploit universality of ϕ_π comparing different corr. functions $\Rightarrow$ estimate of power corrections

Power corrections from the comparison of VV(AA) and PS(SP) correlators



$N_f = 2, \quad 32^3 \times 64, \quad a \simeq 0.071$ fm, $\quad m_\pi \simeq 295$ MeV

- The higher-twist parameter δ_π^2

$$\langle 0|\bar{u}(0)\gamma^\rho ig\widetilde{F}_{\rho\mu}d(0)|\pi(p)\rangle = p_\mu f_\pi \delta_2^\pi; \qquad 0.20 \text{ GeV}^2 \leq \delta_2^\pi \leq 0.25 \text{ GeV}^2$$

- First determination in lattice QCD

$\mathbb{C}$: Quark-antiquark nonlocal operators

Method: X. Ji (LaMET), 1305.1539
A. Radyushkin, 1705.01488

Recent work: I. Cloët *et al.*, 2405.20120, 2407.00206

- Use Wilson line connecting quark fields instead of light-quark propagator

$$\langle 0|\bar{u}(\tfrac{z}{2})\gamma_\mu\gamma_5[\tfrac{z}{2},-\tfrac{z}{2}]d(-\tfrac{z}{2})|\pi(p)\rangle = i f_\pi p_\mu \left\{\int_0^1 dx\, e^{i(x-1/2)pz}[1+O(\alpha_s)]\phi_\pi(x) + \mathcal{O}(z^2\Lambda^2_{\mathrm{QCD}})\right\} \qquad \boxed{z^\mu = (0,0,0,z)}$$

Advantage: Significantly cheaper than $\mathbb{B}$

Disadvantage(s): quark-antiquark separation constrained to be along lines of the lattice
renormalization more complicated
not possible to constrain power corrections

- Emphasis on the calculation of $\phi_\pi(x)$ pointwise in x instead of Gegenbauer coefficients

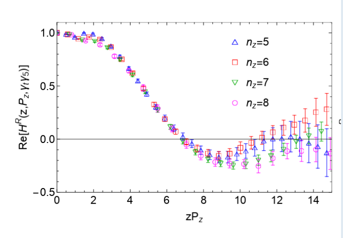
$$i f_\pi \underbrace{\Phi(x)}_{\text{quasi-LCDA}} = \int_{-\infty}^{\infty}\frac{dz}{2\pi}\, e^{-ixpz)}\langle 0|\bar{u}(\tfrac{z}{2})\not{z}\gamma_5[\tfrac{z}{2},-\tfrac{z}{2}]d(-\tfrac{z}{2})|\pi(p)\rangle$$

$$= i f_\pi \underbrace{[1+O(\alpha_s)]}_{\text{matching}}\phi_\pi(x) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{x^2 p_z^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{(1-x)^2 p_z^2}\right) \qquad \Leftarrow \qquad \text{LaMET factorization}$$

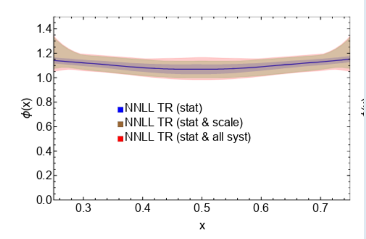
- cannot access $x \to 0$, $x \to 1$, usually assume a fiducial region $0.2 < x < 0.8$

Exploratory studies

— from: I. Cloët *et al.*, 2405.20120, 2407.00206



position-space matrix element

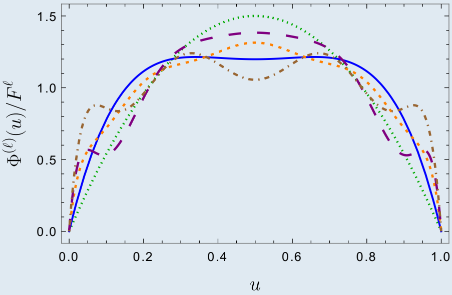


pion LCDA in momentum-fraction space

- Further progress requires larger pion momentum (currently $p_z \leq 2.0 - 2.5$ GeV) and finer lattices (currently $a \geq 0.04$ fm)

Summary: pion LCDA, June 2026

- Reasonable models



solid: RQCD21 [1903.08038]
short dashes: RQCD21+
long dashes: fit to $\gamma\gamma^*\pi$ [1206.3968]
dash-dotted: LAMET [2407.00206]
dots: asymptotic LCDA

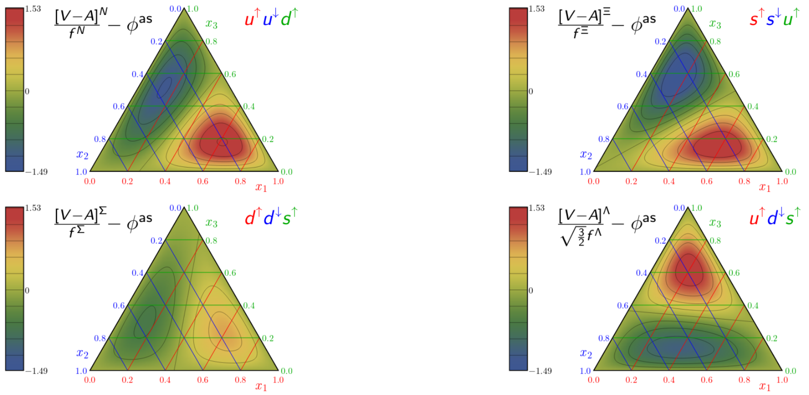
- A sad truth: we do not know how to calculate the inverse moment with defensible error bars

$$\boxed{\int_0^1 dx\,\frac{\phi_\pi(x,\mu)}{1-x} = 3\Big[1 + a_2(\mu) + a_4(\mu) + a_6(\mu) + \dots\Big] = ???}$$

LCDAs of the baryon octet

— from: RQCD, G. Bali *et al.*, 1903.12590

$$\phi_B(x_1,x_2,x_3) = \underbrace{120 x_1 x_2 x_3}_{\text{"asymptotic LCDA"}} \Big[1 + a_{10}(x_1 - x_3) + a_{11}(x_1 - 2x_2 + x_3)\Big]$$

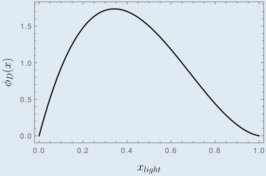


— from: RQCD, G. Bali *et al.*, 1903.12590

Heavy-light mesons (baryons) $B=(b\bar{d}),(b\bar{u})$, $\Lambda_B=(bud)\ldots$

- Two scales:
  Heavy quark mass $m_c \sim 1.4$ GeV, $m_b \sim 4.8$ GeV
  Factorization scale μ (resolution defined by the "hard" scale in the process

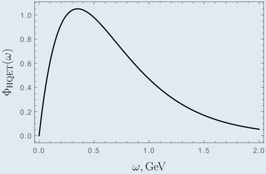
(1) $\mu \gg m_h$ — e.g. D-meson production in processes with $Q^2 \gg m_c^2$



The LCDA is defined in a usual way, gets skewed towards the heavy quark momentum fraction

$$\langle x_{\rm light}\rangle \sim \frac{\Lambda_{\rm QCD}}{m_{\rm heavy}} \qquad \langle x_{\rm heavy}\rangle \sim 1-\frac{\Lambda_{\rm QCD}}{m_{\rm heavy}}$$

(2) $\mu \ll m_h$ — e.g. weak B-decays, $B\to\pi\pi$, $B\to \ell\bar{\nu}\gamma$



$\Phi(\omega,\mu)$: LCDA in the Heavy Quark Effective Theory (HQET)

ω: light-quark energy in the meson rest frame

$$\frac{1}{\lambda_B} = \int_0^\infty \frac{d\omega}{\omega}\Phi(\omega) = ?$$

- LCDAs in HQET: a (relatively) new and important subject

— moments are not defined (no relation to local operators possible

— QCD factorization for $B \to \ell\bar{\nu}\gamma$, wait for experiment

M.Beneke *et al.* 1804.04962

— Exploratory: $B \to \ell\bar{\nu}\gamma$ from lattice QCD

D.Giusti *et al.* 2505.11757

— Exploratory: LaMET-type with matching at $\mu = m_b$

LPC collaboration, H.-F. Gao *et al.* 2604.25902

Take-home message

- LCDAs can be interpreted as the wave functions of valence quarks at zero transverse separation
- LCDAs appear in QCD description of hard exclusive processes, e.g. form factors
- Lattice QCD has become the method of choice to determine LCDAs to phenomenologically relevant accuracy
  Crucial: control over systematic uncertainties
- Several approaches exist. They are complementary and all subject to rapid development:
  — Local operators $\rightarrow$ Moments of LCDAs. In future: gradient flow (?)
  — Nonlocal operators/Euclidean correlation functions. Needed: $a \rightarrow (?) 0.02$ fm, hadron momentum $\rightarrow (?) 5$ GeV
- LCDAs in HQET are gaining attention, important for flavor physics
- Currently no well-defined method exist to calculate the inverse moment,
  which is the most important parameter for phenomenology

Thank you for attention!